A GENERAL FORMULATION OF ANOMALIES IN QED AND UNDERSTANDING OF ARBITRARINESS IN CHIRAL ANOMALY FORMULATIONS AND RENORMALIZATION PRESCRIPTIONS IN PATH-INTEGRAL FORMULATION

by

GAITRI SAINI

DEPARTMENT OF PHYSICS

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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CERTIFICATE

Certified that the work contained in this thesis entitled **"A GENERAL FORMULATION OF ANOMALIES IN QED AND UNDERSTANDING OF ARBITRARINESS IN CHIRAL ANOMALY FORMULATIONS AND RENORMALIZATION PRESCRIPTIONS IN PATH-INTEGRAL FORMULATION"** has been carried out by **Ms. Gaitri Saini** under my supervision and has not been submitted elsewhere for a degree.

SDJoglekar

(Dr. Satish D. Joglekar)
Thesis Supervisor

I.I.T. Kanpur
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Gaitri Saini

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SYNOPSIS

The fact that anomalies may arise in a field theory when it is quantized has been known ever since the ABJ chiral anomaly was discovered in the late sixties using standard perturbation theory techniques.

A derivation of anomalies entirely in the path-integral framework had been lacking until Fujikawa derived almost all the known anomalies in a path-integral formulation. While his derivation of anomalies has had great appeal due to the simplicity and ease with which the anomalies have been derived, there are limitations, which can be attributed to his derivations being somewhat particular and not general enough, as explained below. Fujikawa gives a prescription for defining the path-integral measures in terms of eigenfunctions of the "Energy operators" of the theory, in Euclidean space. By formal, unregularized manipulations he identifies the anomalies with jacobian factors arising in the transformation of these measures under symmetry transformations on the fields. The jacobians which are ill-defined quantities, are regularized by hand using eigenvalues of these energy operators. The particularity of Fujikawa's formalism is due to the crucial dependence of his method on the specific use of energy operator eigenfunctions and eigenvalues. Unacceptable results are got when eigenfunctions and eigenvalues of operators other than "Energy operators" are used. Because of this, many ambiguities in anomaly formulations are not understood satisfactorily within his derivations.

Formulas for anomalies reflect the ambiguities which arise when infinities are regulated. The ambiguities manifest as arbitrariness in anomaly expressions. There are many examples of this. For example, in $(\mathrm{QED})_{2,4}$ one has a family of vector and chiral anomalies. In two-dimensions the family is defined by

$$\langle \partial^{\mu} J_{\mu}{}^{A} \rangle - 2im_{o} \langle \bar{\psi} \gamma_{5} \psi \rangle = \frac{e}{2\pi} (1+a) \, \epsilon^{\mu\nu} \partial_{\mu} A_{\nu}$$

$$\langle \partial^{\mu} J_{\mu}{}^{V} \rangle = \frac{e}{2\pi} (1-a) \, \partial . A$$

The anomalous expressions are arbitrary due to the presence of a real free parameter "a" which also defines the family structure of anomalies. No value of the parameter exists which makes both anomalies vanish. In the formalism as presented by Fujikawa, the family structure remains unexplained, since the presence of the parameter cannot be accounted for.

In the first part of the thesis we have derived the family of anomalies in QED, in two and four-dimensions. This is done by using the complete set of eigenfunctions and the eigenvalues of the parameter dependent hermitian operator $\not{D}_{a} \equiv \not{\partial} + iea\not{A}$, "a" being the real free parameter. In the path-integral, the fermion fields are expanded in terms of the eigenfunctions of $\not{D}_{a}$. In two dimensions for example, we then derive WT identities associated with local chiral and vector transformations and use these to simplify expressions for divergences of vector and axial-vector currents. The anomaly terms that result are unregulated and ill- defined and we regularize them in terms of eigenvalues of $\not{D}_{a}$. This leads to

equations for family of anomalies given above. These same results are also equivalently derived by us, by starting out with definitions of regularized currents $\langle J_\mu^{VM}\rangle$ and $\langle J_\mu^{AM}\rangle$ given as $\langle\bar\psi \exp[-\overleftarrow{\not D}_a^2/M^2]\gamma_\mu \exp[-\not D_a^2/M^2]\psi\rangle$ and $\langle\bar\psi \exp[-\overleftarrow{\not D}_a^2/M^2]\gamma_\mu\gamma_5 \exp[-\not D_a^2/M^2]\psi\rangle$ respectively. This leads to anomaly derivations which are well-defined at all steps of evaluation and no regularizations are introduced by hand at an intermediate stage. It is seen that the anomaly terms get contributions from two sources: a) The jacobian term, *a la* Fujikawa b) A term which can be interpreted as arising from the action. The case a=1 which gives a gauge invariant regularization coincides with Fujikawa's results and the anomaly contribution is entirely from jacobian. The use of the free Dirac opeator $\not\partial$ (a=0 case) is perfectly consistent in our method (unlike Fujikawa's) and here the anomaly comes entirely from action sources.

In a straightforward extension of the formalism given for $(QED)_2$ we derive the equations for the family structure in four dimensions which are based on a <u>single</u> regularized quantity J_μ^{AM}, this being defined in the same way as in 2-dimensions using operator $\not D_a$. Explicit calculations in both two and four dimensions reveal that the free parameter in anomaly equations is related to the free parameter a in $\not D_a$. An important observation is that the arbitrariness in anomaly expressions, manifest through the presence of a parameter, is seen, in the path-integral framework, to be a consequence of the arbitrariness in the definition of the path-integral measure. This arbitrariness in the definition is due to the freedom in expanding the fermion fields in terms of eigenfunctions of a "series" of operators $\not D_a$,

"a" defining the series.

The detailed calculations done in the four dimensional case prove to be quite laborious. We have, in the thesis, given a compact proof for existence family of anomalies in $(QED)_4$ which avoids the need for detailed calculations. This proof is based on examining the parameter dependent contributions to $\langle J_\mu^{AM}\rangle$ and proving the locality of $\langle (J_\mu^{AM} - J_\mu^{AM}|_{a=1})\rangle$, in the limit $M\longrightarrow\infty$. This proof holds, with the appropriate modifications, in two dimensions also.

We have then carried out further generalization of our formulation for anomaly derivations in QED which are based on the axial vector current being regularized in a very general fashion. We define J_μ^{AM} as $\langle\, \bar\psi\; g(\overleftarrow{Y})\gamma_\mu\gamma_5\; f(X)\psi\,\rangle$, where the fermion fields ψ and $\bar\psi$ are expanded in terms of eigenfunctions of fairly general operators X and Y respectively which could depend on many parameters, and very general regularizing functions f and g are used. Using this definition of J_μ^{AM} we show that the <u>one</u> parameter family of anomalies of QED still holds. Thus use of $\not{D}_a$ to define J_μ^{AM} is seen as a special, simple case which is sufficient to yield family structure in 2 & 4 dimensions but its use is not crucial for obtaining family. In fact, this work shows that anomaly derivations in path-integral formulation can be carried out by using a very general basis to define path-integral and regularize J_μ^{A}. We also see that special choices of X, Y, f and g in J_μ^{AM} lead to derivation of anomalies which is equivalent to other known derivations like those based on Feynman diagrammatic calculations, point-splitting calculations, use of non hermitian operators, proper time methods, and Fujikawa's

calculations using $\not{D}$, etc. A derivation of regularized general WT identity is also given in this work.

'Consistent' and 'Covariant' anomalies in non-abelian chiral gauge theories provide another example of ambiguities in anomaly formulations. In these theories one gets either the consistent or the covariant anomaly depending on the procedure of evaluation. In this thesis, we have looked at this aspect of chiral anomaly in a two-dimensional non-abelian gauge theory with the fermionic action given by $S = \int d^2x\ \bar{\psi}_L(i\not{\partial}-\not{A})\gamma_L\psi_L$ in the path-integral formulation. We give definition of the path-integral measure and regularization procedure using a single, hermitian parameter dependent operator $\not{D}_\alpha \equiv \not{\partial} + i\alpha\not{A}$. Explicitly, the fields ψ_L and $\bar{\psi}_L$ are expanded using chiral bases $\{\phi_n{}^L\}$ and $\{\phi_n{}^R\}$ respectively, both of which are sets of eigenfunctions of $\not{D}_\alpha{}^2$. We then give a straightforward regularization for the gauge current using eigen- values of $\not{D}_\alpha{}^2$, in a way analogous to the abelian case, thus defining family of regularized chiral currents. We show that $\alpha = 0$ case gives a consistent current whose covariant divergence gives the consistent anomaly, and the $\alpha=1$ case gives the covariant current whose covariant divergence gives the covariant anomaly.

Therefore, we see this far, that ambiguities in anomaly formulations could be explained, in the path-integral formulation, as arising from the ambiguities in the definition of path-integral measures which are due to the freedom of expanding the fermion fields in any basis. In the concluding part of the thesis we show that renormalization prescription ambiguities can also be correlated with the arbitrariness in the definition of the path

integral measures. Specifically, we discuss the renormalization of bilinear composite operators in path-integral framework at one loop level in the setting of a Yukawa-type theory and show that all ambiguities in their renormalization can be understood as arising from the arbitrariness in the choice of basis for the definition of path-integral.

CHAPTER - I

INTRODUCTION

In a classical field theory, corresponding to every continuous symmetry of the Lagrangian there exists a conservation law given by the Noether's theorem. When the system is quantized, it may happen that the quantization procedure fails to preserve the entire set of such symmetries (as a whole), present at the classical level. The symmetries affected by quantization are termed as "anomalous", and the overall symmetry of the classical system in such a case is reduced because of quantization. The existence of anomalies can be attributed to the unavoidable infinities of relativistic local field theory. In those cases when anomalies are present, no regularization procedures exist which simultaneously respect all the continuous symmetries of the classical theory. Thus, anomalous breaking of symmetries, which is one way in which symmetries are broken, is seen to arise entirely from quantum mechanical effects.

Effects of anomalies show up widely in particle physics. The chiral anomaly was discovered by Schwinger [1]. It has had many consequences of physical interest [2]. When the current associated with the chiral symmetry is not gauged and the related Ward-Takahashi identities are not needed to establish renormalizability of the theory, nonvanishing chiral anomaly can be accomodated in a physical theory where it may lead to observable consequences. For example, it contributes to the $\pi^0 \longrightarrow \gamma\gamma$ decay rate [3]. Here, a good agreement with the

experimental decay rate was obtained with three colours of quarks. This was historically, an early vindication of the concept of colour and QCD. The chiral anomaly was also useful in the elucidation of the U(1) problem in QCD: non-existence of the ninth Goldstone boson associated to a broken symmetry of QCD [4]. When the anomalous current is a gauge source current, there are problems in proving renormalizability of the theory, and for a consistent amd physically interesting theory, anomaly cancellation has to be arranged for. This leads to important results. For example, the requirement that the Standard Model be anomaly free leads to constraints on the particle content of the model. In particular, it leads to the prediction that for every observed lepton there should exist a quark [5].

For a long time, the known derivations of anomalies had been Feynman-diagrammatic, or based on definition of currents regularized by point-splitting techniques [6]. A derivation of anomalies entirely in the path-integral formulation had been lacking until Fujikawa in a series of papers, derived almost all the known anomalies (e.g., chiral anomaly [7-9], trace anomaly [10]), in a simple and straightforward way [11]. Working in a path-integral framework inspired by the one proposed by Fujikawa, we have, in this thesis, given a general formulation of anomalies in QED and an understanding of arbitrariness in chiral anomaly formulations and renormalization prescriptions within the path-integral formulation. To understand the chief motivations and results of the work presented in this thesis it would be useful to examine Fujikawa's path-integral formulation of

anomalies a little more closely. The three main ingredients in his recipe for deriving the anomaly by the functional technique are [11] :-

(1) Definition of the path-integral measure in Euclidean space in terms of the eigenfunctions of the hermitian, covariant "Energy Opertor" of the theory.

(2) Identification of the anomaly with the jacobian factor J, arising in the transformation of the measure under local field transformations. Such an identification is made through a formal derivation based upon manipulations with unregularized quantities.

(3) Regularization of the jacobian factor (which is usually ill-defined), by hand, in terms of the eigenvalues of the 'Energy Operator'.

As an example, consider the derivation of the chiral anomaly in QED by the Fujikawa method [7].

Here the fermion path-integral is given by

$$W[A] = \int D\psi D\bar{\psi}\; e^{\int \bar{\psi}(i\not{D}-m_o)\psi\, d^4x} \quad .$$

Fujikawa defines the path-integral measure $D\psi D\bar{\psi}$ as $\prod_n da_n \prod_m d\bar{b}_m$, where a_n and $\bar{b}_m$ are Grassmann coefficients arising in the expansion of ψ and $\bar{\psi}$ in terms of the complete set of eigenfunctions $\{\phi_n\}$ of $\not{D}$:

$$\begin{aligned} \not{D}\, \phi_n(x) &= \lambda_n\, \phi_n(x) \\ \psi(x) &= \sum_n a_n\, \phi_n(x) \\ \bar{\psi}(x) &= \sum_m \bar{b}_m\, \phi_m^{\dagger}(x) \end{aligned} \tag{1.1}$$

Consider infinitesimal local chiral transformations on the fermion fields ψ and $\bar{\psi}$:

$$\psi(x) \longrightarrow e^{i\alpha(x)\gamma_5}\psi(x) = \sum_n a_n'\phi_n(x).$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}\, e^{i\alpha(x)\gamma_5} = \sum_n \bar{b}_n'\phi_n^{\dagger}(x).$$

The current associated with these transformations is defined in the variation of the action :

$$\delta S = -\int (\partial^{\mu}\alpha(x)) j_{\mu}^{5}(x)\, d^4x - 2im_o \int \bar{\psi}\, \gamma_5\psi\, \alpha(x) d^4x.$$

The transformation of the measure $D\psi D\bar{\psi} \equiv D\bar{b}Da$ is given by

$$Da'D\bar{b}' = Da\, D\bar{b}\, J(\alpha).$$

Here J is the jacobian factor. Considering the chiral transformations as merely change of integration variables in the fermion path-integral $W[A] = \int Da D\bar{b}\, e^{S}$, we would require $\delta W = 0$. This leads to the anomaly equation

$$\langle\, \partial^{\mu} j_{\mu}^{5}(x)\, \rangle = 2im_o \langle\, \bar{\psi}\gamma_5\psi\, \rangle - \frac{\delta}{\delta\alpha} \ln J\ .$$

Since $W[A]$ is not regularized, the derivation of this equation is formal.

Here, $\ln J(\alpha) = -2\int d^4x\; i\alpha(x)\; A(x)$, with $A(x) = \sum_n \phi_n^{\dagger}\gamma_5\, \phi_n = \lim_{x\to y} \operatorname{tr} \gamma_5\delta^4(x-y)$, clearly not well defined. The sum $\sum_n \phi_n^{\dagger}\, \gamma_5\, \phi_n$ is regularized using eigenvalues λ_n of $\not{D}$, as $\sum_n \phi_n^{\dagger}\gamma_5\, \phi_n\, e^{-\lambda_n^2/M^2}$ and evaluated, with the limit $M \longrightarrow \infty$ taken at the end, yielding $\frac{ie^2}{8\pi^2} F_{\mu\nu}\, \tilde{F}^{\mu\nu}$. The regularizing function $e^{-\lambda_n^2/M^2}$ can be replaced

by general functions $f(\lambda_n^2/M^2)$ which satisfy $f(o) = 1$, $f(\infty) = f'(\infty) = \ldots. = 0$.

While Fujikawa's derivation of anomalies in the path-integral formalism has had great appeal due to the simplicity and ease with which almost all kinds of anomalies have been derived, there are drawbacks, which can be attributed to his derivations being somewhat particular and not general enough. The particularity in his formalism is due to the <u>specific</u> use of the "energy operator" eigenfunctions and eigenvalues. Moreover, this choice is also <u>crucial</u> to his method: The formalism does not hold when basis eigenfunctions and eigenvalues of other operators are used. For example, unacceptable results are got when the basis $\{\phi_n\}$ and the set of eigenvalues $\{\lambda_n\}$ belonging to the operator $\not\partial$ are used in defining and regularizing the anomaly factor $\sum_n \phi_n^\dagger \gamma_5 \phi_n$. A null result is obtained for $\lim_{M\to\infty} \sum_n \phi^\dagger \gamma_5 \phi_n \, e^{-\lambda_n^2/M^2}$, which is not surprising, since this is a gauge-field independent quantity. At the same time, the vector anomaly is absent too since the measure $D\psi D\bar{\psi}$ remains invariant under the vector transformations on ψ and $\bar{\psi}$, and there is no jacobian factor. This would imply that we have a regularization respecting chiral symmetry and gauge invariance simultaneously. In other words, the chiral anomaly does not exist !

Due to this particularity of Fujikawa's formalism, many aspects of anomalies are not satisfactorily understood within his path-integral framework. As has been mentioned earlier, anomalies owe their existence to the unavoidable infinities of local, relativistic Quantum Field Theory. The need to regulate these

infinities introduces ambiguities in anomaly formulations. These ambiguities manifest as arbitrariness in anomaly expressions. There are many examples of this. For example:

(1) In two and four dimensional QED one has the "family" of chiral and vector anomalies. This family implies that no regularization procedures exist, which respect chiral symmetry (up to mass terms) and gauge invariance at the same time. In two dimensions, this family structure is neatly exhibited in the following pair of equations which are in terms of the divergences of axial-vector and vector currents [12-17] :-

$$\langle \partial^{\mu} J^{A}_{\mu} \rangle = 2im_{o} \langle \bar{\psi} \gamma_{5} \psi \rangle + \frac{e}{2\pi} (1+a)\, \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu} \ . \qquad (1.2a)$$

$$\langle \partial^{\mu} J^{V}_{\mu} \rangle = \frac{e(1-a)}{2\pi}\, \partial^{\mu} A_{\mu} \ . \qquad (1.2b)$$

The anomalous terms are arbitrary due to the presence of a free parameter 'a', which defines the family structure and reflects the chamaleon like character of the anomaly: There is no value of parameter for which the anomalous terms from both the axial-vector current and vector current divergence vanish. Such a family is, in fact, the <u>essence</u> of the chiral anomaly, which is identified with the case a = 1 in Eqs. (1.2), by invoking gauge invariance. Such a family structure exists in four-dimensions too [6, 18-20].

(2) In non-abelian gauge theories one gets either the 'consistent' or 'covariant' anomaly depending on the procedure of evaluation [21].

All such ambiguities are not satisfactorily understood

within Fujikawa's path-integral framework. For example, the family structure remains unexplained. In the first part of the thesis we have derived the family structure of anomalies in two and four dimensional QED, in the path-integral formulation. In chapters II and III, we have derived the family of anomalies in two-dimensional QED [13,14]. This has been done by making use of the parameter dependent operator $\not{D}_a = \not{\partial} + iea\not{A}$, which is hermitian in Euclidean space. ('a' is the free parameter). We have shown that as far as the family structure is concerned, it is sufficient to consider $\not{D}_a$ as the operator whose complete set of eigenfunctions are used to expand the fermion fields and whose eigenvalues are used for regulating ill-defined quantities. The free parameter 'a' in this operator is the same as the free parameter defining the family equations. More importantly, it has also been shown in these chapters that contributions to the anomaly come from two sources: (a) from a jacobian term, *a la* Fujikawa and (b) from terms which can be interpreted as arising from the action. This result is arrived at by careful manupulations with well-defined regularized quantities in Chapter III. (In the absence of regularization, anomalous terms from the action would be formally zero, but a properly regularized approach shows that these terms contribute, and are important in establishing family structure). The case a=1, i.e, the use of $\not{D}$ to derive anomaly, leads to results which coincide exactly with Fujikawa's derivation of the chiral anomaly: The vector current is anomaly free (naturally, since $\not{D}$ defines a gauge-invariant regularization) and the axial-vector current divergence has the full anomaly (the usual chiral anomaly), which comes entirely from

the jacobian. The case a=0, amounting to the use of the free Dirac operator $\not\partial$ has anomaly contributions coming entirely from the action sources. The jacobian does not contribute anything. (In Fujikawa's derivation, use of $\not\partial$ was disallowed altogether, since as shown earlier, it led to unacceptable results).

In chapter II, we have derived these results in the context of $(QED)_2$ in the following steps [13]. The fermion fields ψ and $\bar{\psi}$ are expanded in the basis of eigenfunctions of $\not{D}_a$. W-T identities associated with local chiral and vector transformations are derived, and these are used to simplify the expressions for divergences of vector and axial-vector currents. The anomaly terms that result are ill-defined and unregularized. We give a prescription for regularizing them in terms of eigenvalues of $\not{D}_a$. This leads, then, to the equations of family of anomalies. Other attempts to derive the family structure in two dimensions have been made [15-17]. In all these works, anomaly is always assumed to arise from jacobian factors of some kind (an assumption which, as we shall show, is erroneous and not true generally). We compare our derivation with these works, in this chapter.

In chapter III, we derive the results of chapter II by starting out with definitions of regularized axial-vector and vector currents, J_μ^{AM} and J_μ^{VM} respectively [14]. We directly evaluate $\lim\limits_{M\to\infty} \langle \partial^\mu J_\mu^{AM} \rangle$ and $\lim\limits_{M\to\infty} \langle \partial^\mu J_\mu^{VM} \rangle$ in the path-integral formulation. J_μ^{AM} is defined as

$$\langle J_\mu^{AM} \rangle = \langle \, \bar{\psi} \; e^{-\not{D}_a^{\dagger 2}/M^2} \gamma_\mu \gamma_5 \; e^{-\not{D}_a^2/M^2} \psi \, \rangle \, .$$

J_μ^{VM} is similarly defined. Expansion of ψ and $\bar{\psi}$ in the basis

of $\not{\phi}_a$ is assumed. The whole procedure of anomaly derivation turns out to be well-defined at all steps, in that the regularizing factors arise in a natural manner and no *ad-hoc* regularizations have to be introduced by hand at an intermediate stage of derivation. The details of calculations are the same as those in chapter II. In canonical operator formalism we have definitions of regularized currents given by point-splitting techniques. [6,12]. The definitions of regularized currents that are given in this chapter, provide an analogue to these regularized current definitions, in the path-integral framework.

Our derivations of the family of anomalies in 2-dimensions are seen to have a very natural extension in four-dimensional QED (unlike, say, Refs.16,17). We have carried out such an extension in chapter IV. [19]. Expressions for family of anomalies in 4-dimensions can be found in standard texts [see, for example, Ref. 18], though no such derivations had previously been done in the path-integral formalism. Talking in the Feynman diagrammatic language, family of anomalies can be expressed in terms of the three-point function of electrodynamics, $T_{\mu\nu\lambda}(k_1,k_2,q)$ [18]:

$$T_{\mu\nu\lambda} = i\int d^4x_1\, d^4x_2\, \langle 0|T[J^V_\mu(x_1)\, J^V_\nu(x_2)\, J^A_\lambda(0)]|0\rangle\, e^{ik_1.x_1 + ik_2.x_2}.$$

The equations defining the family structure are :-

$$q^\lambda T_{\mu\nu\lambda}(\beta) = 2mT_{\mu\nu}(0) - \frac{(1-\beta)}{4\pi^2}\, \varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma\, k_2^\rho \qquad (1.3a)$$

and

$$K_1^\mu\, T_{\mu\nu\lambda}(\beta) = \frac{(1+\beta)}{8\pi^2}\, \varepsilon_{\nu\lambda\sigma\rho}\, k_1^\sigma\, k_2^\rho\, , \qquad (1.3b)$$

where

$$T_{\mu\nu}(k_1,k_2,q,\beta)=i\int d^4x_1 d^4x_2 \; e^{ik_1.x_1+ik_2.x_2}\langle 0|T[J^V_\mu(x_1)J^V_\nu(x_2)\bar\psi(0)\gamma_5\psi(0)]|$$

Such a family is a manifestation of the fact that $T_{\mu\nu\lambda}$ is ambiguous because it has, in one-loop order, a (superficially) linearly divergent diagram and as such, shift of the loop momentum (integration variable) is not legitimate [6]. This leads to the existence of a free parameter β in the definition of $T_{\mu\nu\lambda}$, which reflects the arbitrariness in the choice of loop momentum. Talking in the language of currents, we derive the following analogues of equations (1.3) [19] :

$$\partial^\nu W_\nu[A] = \frac{ie^2(1-\beta)}{16\pi^2} F_{\mu\nu}\tilde F^{\mu\nu} + 2im_o W_P[A], \tag{1.4a}$$

$$\partial^\mu_y \frac{\delta W_\nu[A(x)]}{\delta A_\mu(y)} = \frac{ie^2(1+\beta)}{16\pi^2} \varepsilon_{\nu\mu\rho\sigma} F^{\mu\rho}(x) \partial^\sigma_x \delta^4(x-y). \tag{1.4b}$$

Here $W_\nu = \lim_{M\to\infty} \langle J_\nu^{AM} \rangle$. J_ν^{AM} is as defined in the two dimensional context, using operator $\not{D}_a$.

$$W_P = \lim_{M\to\infty} \langle \bar\psi \; e^{-\overleftarrow{\not{D}}_a^2/M^2} \gamma_5 \; e^{-\not{D}_a^2/M^2} \psi \rangle .$$

It is shown that the parameter β appearing in equations (1.4) is related to the parameter 'a' in $\not{D}_a$ through the relation

$$\beta = -a^2 . \tag{1.5}$$

The calculations done in chapter 4 for deriving anomaly equations are detailed and laborious. In chapter V of the thesis

we give a compact argument to prove the family structure of anomalies in $(QED)_4$ which avoids the necessity of doing detailed calculations [20]. The proof is based on an examination of W_ν (which is as defined above) and its properties. In particular, it is shown that family of anomalies of Eqs. (1.4) follows if $(W_\nu - W_\nu|_{a=1})$ is local with β being related to the parameter 'a' of $\not{\phi}_a$ by

$$\beta = -1 + \frac{i8\pi^2 f(a)}{e^2} . \tag{1.6}$$

$f(a)$ is some function satisfying $f(a)|_{a=1} = 0$ [Its exact form is, of course, not determined without explicit calculation of chapter IV]. This proof holds, with the appropriate modifications, in two dimensions also. Such arguments are of great use when we carryout further generalization of our derivations of anomalies in QED [22] in chapter VI and also when we discuss 'covariant' and 'consistent' anomalies in a two dimensional non-abelian gauge-theory [23] in chapter VII.

So far, the derivation of anomalies and their family structure made use of the specific operator $\not{\phi}_a$ and specific regularizing functions to define the path-integral and regulate ill-defined quantities. However, as seen from a general view-point, the path-integral can be defined very generally, in that, the fermion fields can be expanded in <u>any</u> bases in general. (For example, ψ could be expanded in terms of any complete set of basis functions which could be taken to be eigenfunctions of a general hermitian operator X. $\bar{\psi}$ too could be similarly defined).

Therefore, an important result like derivation of anomalies and their family structure should not, in our view, depend crucially on specific choices of operators (like for example $\not{D}_a$) to define the path-integral. This basic observation has, at the outset, motivated the work done in chapter VI where we show that derivation of anomalies can in fact be carried out using very general definition of the path-integral and associated ill-defined quantities. In this chapter, we give a very general formulation of anomalies in QED which is based on definition of the axial-vector current regularized in a very general manner [22] :

$$\langle J_\mu^{AM} \rangle \;=\; \langle\, \bar\psi\, g(\overleftarrow{Y})\, \gamma_\mu \gamma_5\, f(X)\, \psi \,\rangle \;. \qquad (1.7)$$

The fermion fields ψ and $\bar\psi$ are expanded in bases of eigenfunctions of fairly general operators X and Y respectively, which could depend on many parameters. f and g are very general regularizing functions. Using this definition of J_μ^{AM} we show that the one parameter family of anomalies given by Equations (1.4) still holds with

$$\beta \;=\; -1 + \frac{i8\pi^2 \mathbb{F}}{e^2} \quad . \qquad (1.8)$$

$\mathbb{F}$ is some function which depends in particular, on the parameters in X and Y. The proof of the family structure is an extension of the one given in chapter V for the particular case $X = Y = \not{D}_a^{\,2}/M^2$ and $f = g = e^{-X}$. This proof holds, provided certain assumptions on the form of X, Y, f and g are made and these are discussed at length in this chapter. We also show that special

choices of X, Y, f and g in the definition of J_{μ}^{AM} leads to derivation of anomalies which is equivalent to other known derivations like those based on Feynman diagrammatic calculations, point-splitting calculations, proper time methods, Fujikawa's calculations using $\not{D}$, etc. We therefore see that anomalies can be derived in the path-integral formulation with a very wide choice of bases to define the path-integral. In the course of evaluating anomalies in a rigorous fashion using general bases, we also understand that ambiguities in anomaly formulations (which manifest as arbitrariness in anomaly expressions) can be fully understood as arising from the arbitrariness in defintion of the path-integral. In chapters II-V, we have already seen that the arbitrariness, given by the presence of the free parameter in anomaly expressions, could be seen to arise in path-integral framework, from the arbitrariness in the choice of basis to define the path-integral. (The parameter 'a' in $\not{D}_a$ provided the arbitrariness in choice of operator and basis functions). In this chapter, this understanding of arbitrariness in anomaly formulations in path-integral formulation acquires a wide perspective, in light of the generalized formulation of anomalies based on a full mathematical treatment, presented here. In this chapter we have also given a derivation of general, regularized W-T identity. Using this regularized identity it is generally shown that contributions to the anomaly come from both the jacobian and action sources in the path-integral. (This was seen in earlier chapters in specific examples, where use of operator $\not{D}_a$ was made). It is shown that contributions to anomaly from action sources arise whenever use of bases other than the basis of

"Energy Operator" of the theory is made, and the anomaly derivation is based upon manipulations with <u>properly</u> <u>regularized</u> quantities. Such action contributions are present even when bases of gauge covariant operators are used.

Consistent and covariant anomalies in non-abelian gauge theories provide another example of ambiguities in anomaly formulations. In these theories one gets either the 'consistent' or the 'covariant' anomaly depending on the procedure of evaluation [see for example Ref. 21]. The 'consistent' anomaly is so named since it satisfies certain consistency conditions, the Wess-Zumino consistency conditions [24]. This anomaly appears in the covariant divergence of a 'consistent current' which is defined by the gauge variation of the vacuum functional. For the non-singlet, non-abelian chiral anomaly, the consistency conditions restrict the anomaly from having a covariant form. On the other hand, anomaly in the gauge current may be obtained via a regularization that is gauge covariant. Such an anomaly is called the 'covariant' anomaly and the associated current is the covariant current. In chapter VII of this thesis, we look at this aspect of the chiral anomaly in a two-dimensional non-abelian gauge theory with the fermionic action given by $S=\int d^2x\bar{\psi}_L(i\not\partial-\not A)\gamma_L\psi_L$ in the path-integral formulation [23]. By making use of a parameter dependent hermitian operator $\not D_\alpha = \not\partial + i\alpha\not A$ (α is a free parameter), we give definition of a one-parameter family of regularized gauge currents and then show that special choices of the parameter lead to covariant and consistent currents, whose covariant divergences give the covariant and consistent anomalies

respectively. This approach of deriving the covariant and consistent currents in the path-integral approach is novel, and an interesting application of parameter dependent regularizations in context of the non-abelian theory. Explicitly, we have done the following. We define a family of regularized chiral currents as

$$\langle J_\mu{}^{AM}(x)\rangle = \langle\bar{\psi}_L(x)\exp(-\bar{\not{D}}_\alpha{}^2/M^2)\ \frac{\lambda^a}{2}\ \gamma_\mu\gamma_L\ \exp(-\not{D}_\alpha^2/M^2)\psi_L(x)\rangle, \tag{1.9}$$

in a way analogous to the abelian case. In a straightforward path-integral prescription, the fields ψ_L and $\bar{\psi}_L$ are expanded in chiral bases $\{\phi_n{}^L\}$ and $\{\phi_n{}^R\}$ respectively, both of which are sets of eigenfunctions of $\not{D}_\alpha{}^2$. The regularization is provided in terms of the eigenvalues of this operator. We show that $\alpha = 0$ gives a consistent current and $\alpha = 1$ gives the covariant current. We further see that contributions to anomaly arise from both the jacobian and action sources in the path-integral. In particular, it is seen that covariant anomaly comes entirely from regularized jacobian sources and consistent anomaly comes entirely from regularized action sources. Other attempts to derive consistent or covariant anomalies in the context of path-integral framework have been made. All these works are based on the identification of anomaly with jacobian (an erroneous assumption, generally, as we show in the thesis). All kinds of regularizing operators and regularizations of the jacobians are played around with to yield either the covariant anomaly (see, for example, Ref. 25) or the consistent anomaly (see, for example, Ref. 26). We compare our work with these works and show how our approach is more natural and straightforward as compared to theirs. Of course, in none of

these works, both covariant and consistent anomalies are seen to arise so naturally from a single regularized quantity, as in our case. By noting that, selecting different values of parameter α in the regularized current of Eq. (1.9), gives us different currents whose divergences lead to different anomalies, we again correlate the arbitrariness in anomaly formulations in the path-integral formulation, with the arbitrariness in the choice of basis for defining the path-integral.

In chapter VIII, in the concluding part of thesis, we show that, not just ambiguities in anomalies, but general renormalization prescription ambiguities can also be correlated with the arbitrariness in the definition of the path-integral [27]. Specifically, we discuss the renormalization of bilinear composite operators within our path-integral framework at one-loop level in the setting of a Yukawa-type theory, and show that all ambiguities in their renormalization can be understood as arising from the arbitrariness in choice of basis for definition of path-integral.

Parameter dependent operators, used for showing the results in this thesis, have been used in other contexts also, like, for example, to define the Chiral Schwinger Model (CSM) in a consistent fashion [28-31].

CHAPTER - II

FAMILY OF ANOMALIES IN $(QED)_2$ IN PATH-INTEGRAL FORMULATION

2.1 INTRODUCTION

Various aspects of the chiral anomaly have been extensively studied in two-dimensional fermionic abelian gauge theories [12-17,28-31,32,33]. From the point of view of study of anomalies, two dimensional models have been of great pedagogical value. Many aspects of the anomaly which are present in realistic four-dimensional models, (such as family structure of anomalies in QED) are revealed by these simple 2-d theories and these have hence served as "testing grounds" for developing many an idea on anomaly formulations.

In theories where fermions are coupled to vector gauge fields (massive or massless QED for example), the existence of the chiral anomaly can be traced to the incompatibility of the quantization procedure with the simultaneous realization of gauge symmetry and chiral symmetry (upto mass terms), though both symmetries hold at the classical level. Quantization introduces the need for regularization, and no regularization procedure exists which respects both symmetries at the same time. This result was established originally in the two-dimensional context by K. Johnson [32]. It can be seen by detailed calculations that the divergence (log) in one-loop graph contributing to the two current correlation function $\langle j_5^{\mu}(x) j^{\nu}(y) \rangle$, in a dynamically non-trivial theory, cannot be regulated so that both the vector and axial-vector vertices are conserved. Therefore, the basic

thing, really, is not that there is a chiral anomaly alone, but that there exists a "family" of chiral and vector anomalies. This family structure can, in $(QED)_2$ for example, be given by the pair of equations involving the divergences of the vector and axial-vector currents [13,14][1]

$$\langle \partial^{\mu} J^{A}_{\mu} \rangle = 2im_{o} \langle \bar{\psi}\, \gamma_5 \psi \rangle + \frac{e}{2\pi}(1+a)\, \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu} \tag{2.1a}$$

$$\langle \partial^{\mu} J^{V}_{\mu} \rangle = \frac{e(1-a)}{2\pi}\ \partial . A \tag{2.1b}$$

Here 'a' is a real free parameter which defines the family structure and reflects the arbitrariness in the anomalous expressions. No value of the parameter exists, such that both current divergences are anomaly free. The vector current divergence can have non-zero value, *a priori*. Of course, the physical requirement of gauge invariance fixes the value a=1 and then Eq. (2.1) gives us the familiar chiral anomaly. This family of anomalies has been derived in conventional canonical operator formalism using parameter dependent definitions of currents, regularized by point-splitting techniques [6,12,34].

For a long time a direct treatment of anomalies in the path- integral formulation had been lacking, though the formulation was widely used, especially in quantizing gauge theories. This was until Fujikawa derived the various known anomalies in the path-integral formalism [11]. The path-integral measures in such formulations are not invariant under appropriate

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[1] See also Refs. 12, 15-17, for the family in the massless case.

field transformations, leading to nontrivial jacobian factors, which are the sources of anomalies.

In general, in these treatments, one expands the fields in terms of the eigenfunctions of the Euclidean "energy-operators" [7], regularizes the Jacobian in terms of the eigenvalues of them and it is this regularization that is responsible for the known anomalies. For example, for the case of the chiral anomaly, one expands the fermion fields ψ and $\bar{\psi}$ in terms of the eigenfunctions of the hermitian operator $\not{D}$, defines $D\psi D\bar{\psi}$ as $\prod_n d\bar{b}_n \prod_n da_n$ [7], obtains the jacobian for the infinitesimal local chiral transformations which contains an ill-defined quantity $\sum_n \Phi_n^\dagger(x)\gamma_5\Phi_n(x)$, and this is regularized in terms of the eigenvalues λ_n of $\not{D}$ as $\sum_n \Phi_n^\dagger(x)\, \gamma_5\, \Phi_n(x)\, e^{-\lambda_n^2/M^2}$. When the latter is evaluated in the limit $M \to \infty$, this leads to the standard chiral anomaly, and a similar treatment leads to no vector anomaly.

The question was as to how to establish the family structure such as given in Eqs. (2.1) in the path-integral framework. Equivalently, how do we incorporte the arbitrary parameter in the Fujikawa approach ? A lot of works have addressed this problem [13-17]. The basic difficulty is that in the formalism as presented by Fujikawa, it is not possible to understand the existence of the vector anomaly at all, since the path-integral measure $D\psi D\bar{\psi}$ is necessarily invariant under the infinitesimal local vector transformation (see Eq. (2.22) below) on ψ and $\bar{\psi}$ and there is no jacobian factor. Further, Fujikawa makes specific use of Euclidean "Energy operator" $\not{D}$ to define the path-integral measure and regularization procedure. This choice is crucial to

his method, since use of any other operator ($\not\partial$ for example) leads to unacceptable results. (see page 5, chapter I).

We approached the problem [13,14] by making a basic observation. In general, it should be possible to expand ψ in terms of any complete set of basis functions $\{\phi_n\}$. For concreteness, one could choose this to be the set of eigenfunctions of any Hermitian operator X and try to derive the anomaly. It would be technically difficult to proceed with the evaluation of the anomaly unless the operator is a "generalization" of the operator $\not\partial$ used in the plane-wave basis.

In this work, we use the Hermitian operator $\not D_a = \not\partial + iea\not A$ which is a generalization of $\not D$. "a" is a real continuous parameter and its range of values over the real line defines a class of operators. In particular a=0 gives the free Dirac operator $\not\partial$ and a=1 gives the gauge covariant $\not D$. The eigenvalues of the operator $\not D_a$ ($a \neq 1$) are not gauge- invariant and any regularization in terms of these eigenvalues is not going to preserve gauge-invariance. Hence, at the outset, one knows that the vector current anomaly is not likely to vanish and hence the axial-vector current anomaly is not likely to be the same. So, one could (at best) expect to obtain a family of vector and axial-vector anomalies. We work in the context of two dimensional QED. We give a treatment complete in all detail for the derivation of the family of vector and chiral anomalies.

The general viewpoint adopted by Fujikawa [7] is that the anomaly comes entirely from the regularized Jacobian, and it does so in his regularization. In the Feynman-diagrammatic

method, anomaly effectively comes from the action. In general, there is no reason why the anomaly could not arise from both sources. In fact, we find this in our treatment of the regularization using the operator $\not{D}_a$.

In Sec. (2.2), we define our notations. In Sec. (2.3), we derive the W-T identities for local chiral and vector transformations. In Sec. (2.4), we evaluate the divergences of axial-vector and vector currents, and use the W-T identities to simplify them. We find that in addition to the usual Jacobian factor, there is an additional term. Both the terms are unregularized and ill-defined.We give a prescription for their regularization. We note that the additional term arises from the action. In sec. (2.5), we evaluate the anomaly and obtain the family of anomalies of equations (2.1). In section (2.6) we make some concluding remarks on our procedure and results and compare our work with other works addressing the same problem, viz., Refs. 16,17 and in particular Ref. 15.

2.2 PRELIMINARY

We consider the two-dimensional Euclidean Lagrangian for a Dirac fermion ψ :

$$\mathcal{L} = \bar{\psi} \, (i \not{D} - m_o) \, \psi$$

where $\not{D} = \gamma^\mu D_\mu = \gamma^\mu \, (\partial_\mu + ieA_\mu)$, and A_μ is a real abelian external field. The γ-matrices γ_1 and γ_4 are antihermitian, which makes $\not{D}$ a hermitian operator in Euclidean space. The metric in Euclidean space is $g_{\mu\nu} = (-1, -1)$; μ, $\nu = 1,4$. The γ-matrices satisfy

$$\gamma^{\mu}\,\gamma^{\nu} = g^{\mu\nu} + \varepsilon^{\mu\nu}\,\gamma_5, \tag{2.2}$$

with $\varepsilon^{\mu\nu} = -\,\varepsilon^{\nu\mu}$ and $\varepsilon^{14} = -\,i$. $\gamma_5 \equiv +\,i\,\gamma^1\gamma^4$ is hermitian and satisfies $\gamma_5{}^2 = 1$.

We define the path-integral in terms of the basis of the complete set of eigenfunctions of a hermitian operator following Fujikawa [7]. We shall, however, choose this hermitian operator not to be $\not{D}$ but $\not{D}_a \equiv \not{\partial} + iea\not{A}$, where a is a real, continuous parameter [15,33]. We let

$$\begin{aligned} \not{D}_a\,\phi_n\,(x) &= \lambda_n\,\phi_n\,(x), \\ \phi_n^{\dagger}\,\overleftarrow{\not{D}}_a &= \lambda_n\,\phi_n^{\dagger}, \end{aligned} \tag{2.3}$$

where λ_n are real. We expand

$$\begin{aligned} \psi(x) &= \sum_n a_n\,\phi_n\,(x), \\ \bar{\psi}\,(x) &= \sum_n \phi_n^{\dagger}\,(x)\,\bar{b}_n, \end{aligned} \tag{2.4}$$

and define the measure in the path-integral

$$D\bar{\psi}\,D\psi = \prod_m d\bar{b}_m \prod_n da_n. \tag{2.5}$$

The path integral W [A] is

$$\begin{aligned} W[A] &= \int D\psi\,D\bar{\psi}\,e^S = \int \prod_m d\bar{b}_m \prod_n da_n\,e^S \\ &\equiv \int D\bar{b}\,Da\,e^S, \end{aligned} \tag{2.6}$$

with S expressed in terms of a_n and $\bar{b}_m$. It reads

$$S = \sum_p \sum_q \bar{b}_p\,a_q\,\xi_{pq}, \tag{2.7}$$

$$\text{where} \quad \xi_{pq} \equiv (i\lambda_q - m_o)\,\delta_{pq} - e(1-a)\int d^2x\,\phi_p^{\dagger}\,\not{A}\,\phi_q. \tag{2.8}$$

Here neither the action nor the generating functional are regularized. Consequently, manipulations done in next section with these quantities, like derivation of W-T identities related to transformations on the fermion fields, would be formal. The W-T identity for the chiral transformations, for example, (see equations (2.9) and (2.20) below) involves unregularized terms. However, as we shall see later in chapter VI, section (6.5), we could derive the W-T identity in a manner where all terms are naturally well-defined and regularized, by considering modified form of infinitesimal chiral transformations [22]. The same thing could be done for the vector case too. In the next section however, we shall be content with manipulations at a formal level. (Or, we may tentatively assume a finite-mode cut-off regularization, such that, for example, the sum in Eq. (2.4) and products in (2.6) go from n = 1 to N).

In the second part of this work [14] which is described in Chapter III, we take up the question of defining the currents themselves in a regularized form along the lines suggested by Verstegen [36], and give meaning to the entire procedure at every step.

With these remarks in mind we go ahead to the next section.

2.3 W-T IDENTITIES

In this section, we shall obtain the W-T identities for W[A] of Eq. (2.6) from local chiral and vector transformations. These W-T identities will, of course, involve a Jacobian. Also terms will arise from the change in the action, but as seen in

section (2.4), these terms will <u>not</u> coincide with a term of the kind $\int d^2x\ \partial^\mu\ \alpha(x)\ J_\mu$ (where J_μ is the corresponding current), contrary to what may be expected. This is explained in a greater detail in section (2.4).

A. Chiral transformations

We make infinitesimal local transformations:

$$\begin{aligned} \psi' &= (1+i\alpha(x)\gamma_5)\psi \equiv \sum_n a'_n\, \phi_n\,, \\ \bar{\psi}' &= \bar{\psi}\,(1+i\alpha(x)\gamma_5) \equiv \sum_n \bar{b}'_n\, \phi^\dagger_n\,. \end{aligned} \tag{2.9}$$

These lead to transformations on a_n and $\bar{b}_n$ as

$$\begin{aligned} a'_n &= a_n + \sum_m C_{nm}\, a_m\,, \\ \bar{b}'_n &= b_n + \sum_m \bar{b}_m\, C_{mn}\,, \end{aligned} \tag{2.10}$$

with

$$C_{mn} = \int \phi^\dagger_m(x)\ [i\alpha(x)\gamma_5]\ \phi_n(x)\ d^2x\ . \tag{2.11}$$

Now consider

$$W[A] = \int D\bar{b}\ Da\ \ e^{S[a,\bar{b}]}, \tag{2.12}$$

and make the change of variables of Eq. (2.10) in W[A] to obtain

$$W[A] = J(\alpha) \int D\bar{b}' Da'\ e^{S'[a',b']}, \tag{2.13}$$

where

$$D\bar{b}\ Da = J(\alpha)\ D\bar{b}'\ Da', \tag{2.14}$$

and

$$\begin{aligned} S'[a',\bar{b}'] &= S[a,\bar{b}] \\ &= S[a',\bar{b}'] + \Delta S[a',\bar{b}']. \end{aligned} \tag{2.15}$$

One then has

$$W[A] = \int D\bar{b}'\, Da'\; e^{S[a',\bar{b}'] + \Delta S\,[a',\bar{b}'] + \ln J(\alpha)}$$

$$= \int D\bar{b}\, Da\;\; e^{S[a,\bar{b}] + \Delta S\,[a,\bar{b}] + \ln J(\alpha)}. \tag{2.16}$$

A comparison of Eqs (2.12) and (2.16) gives the W-T identity

$$\int D\bar{b}\, Da\;\; e^{S[a,\bar{b}]}\, [\Delta S\,[a,\bar{b}] + \ln J(\alpha)] \;=\; 0. \tag{2.17}$$

A straightforward calculation shows that, to first order in α,

$$\Delta S\,[a,\bar{b}] \;=\; -\sum_p \sum_q \sum_n \bar{b}_p\, a_q\, (C_{pn}\, \xi_{nq} + C_{nq}\, \xi_{pn}), \tag{2.18}$$

and

$$\ln J(\alpha) \;=\; 2 \sum_n C_{nn}\,, \tag{2.19}$$

and, thus, Eq (2.18) leads to the W-T identity

$$\langle \sum_p \sum_q \sum_n \bar{b}_p\, a_q\, (C_{pn}\, \xi_{nq} + C_{nq}\, \xi_{pn}) - 2 \sum_n C_{nn} \rangle = 0, \tag{2.20}$$

where

$$\langle 0 \rangle \equiv \frac{1}{W[A]} \int D\bar{b}\, Da\;\; e^{S}\, 0\,. \tag{2.21}$$

B. Vector Transformations

A similar procedure applied to the vector transformations,

$$\begin{aligned} \psi' &= (1 + i\beta(x))\, \psi, \\ \bar{\psi}' &= \bar{\psi}\, (1 - i\beta(x)), \end{aligned} \tag{2.22}$$

leads to the W-T identity

$$\langle \sum_p \sum_q \sum_n \bar{b}_p\, a_q\, (d_{pn}\, \xi_{nq} - d_{nq}\, \xi_{pn}) \rangle \;=\; 0, \tag{2.23}$$

where $\quad d_{pn} \equiv \int \phi_p^{\dagger}(x)\ [\ i\beta(x)]\ \phi_n(x)\ d^2x.$

Using the expressions for ξ_{pq} from Eq. (2.8) the W-T identities of Eqs. (2.20) and (2.23) can be further simplified to

$$\langle \sum_p \sum_q \bar{b}_p\, a_q\, c_{pq}\, [(i\lambda_q - m_o) + (i\lambda_p - m_o)]$$

$$- e(1-a) \sum_p \sum_q \sum_n \bar{b}_p\, a_q\, [c_{pn} \int \phi_n^{\dagger} \not{A}\, \phi_q\, d^2x + c_{nq} \int \phi_p^{\dagger} \not{A}\, \phi_n\, d^2x]$$

$$- 2 \sum_n c_{nn} \rangle = 0, \tag{2.24}$$

and

$$\langle \sum_p \sum_q \bar{b}_p\, a_q\, d_{pq}\, (i\lambda_q - i\lambda_p)$$

$$-e(1-a) \sum_{pqn} \bar{b}_p\, a_q\, [d_{pn} \int \phi_n^{\dagger} \not{A}\, \phi_q d^2x - d_{nq} \int \phi_p^{\dagger} \not{A}\, \phi_n\, d^2x] \rangle = 0. \tag{2.25}$$

2.4 DIVERGENCES OF CURRENTS

In this section we shall obtain expressions for the divergences of the vector and axial-vector currents $\langle \partial^{\mu} J_{\mu}^{V} \rangle$ and $\langle \partial^{\mu} J_{\mu}^{A} \rangle$, and simplify using the W-T identities obtained in section (2.3). We define

$$J_{\mu}^{A} = \bar{\psi}\, \gamma_{\mu}\, \gamma_5\, \psi\,,$$

$$J_{\mu}^{V} = \bar{\psi}\, \gamma_{\mu}\, \psi\,, \tag{2.26}$$

and obtain expressions for $\partial^{\mu} J_{\mu}^{A}$ and $\partial^{\mu} J_{\mu}^{V}$. A straightforward calculation shows that

$$\int d^2x\ \alpha(x)\ \partial^\mu J^A_\mu(x) - 2im_o \int d^2x\ \alpha(x)\ \bar\psi\ \gamma_5\ \psi$$

$$= \sum_p \sum_q \bar b_p\ a_q\ (i\lambda_p - m_o) c_{pq} + \sum_p \sum_q \bar b_p\ a_q\ (i\lambda_q - m_o) c_{pq}\ , \qquad (2.27)$$

and

$$\int d^2x\ \beta(x)\ \partial^\mu J^V_\mu = \sum_p \sum_q \bar b_p\ a_q\ (i\lambda_p - i\lambda_q)\ d_{pq}. \qquad (2.28)$$

Using the W-T identities of Eqs. (2.24) and (2.25) we obtain

$$\int d^2x\ \alpha(x)\ \langle\ \partial^\mu J^A_\mu\ (x) - 2im_o\ \bar\psi\ \gamma_5 \psi\ \rangle$$

$$= e(1-a) \langle \sum_p \sum_q \sum_n \bar b_p a_q [c_{pn} \int \phi^\dagger_n \not{A}\ \phi_q\ d^2x + c_{nq} \int \phi^\dagger_p \not{A}\ \phi_n d^2x]\ \rangle$$

$$+\ 2 \sum_n c_{nn}\ , \qquad (2.29)$$

and

$$\int d^2x\ \beta(x)\ \langle \partial^\mu J^V_\mu\ (x) \rangle = -e(1-a)\ \langle \sum_p \sum_n \sum_q \bar b_p a_q [d_{pn} \int \phi^\dagger_n \not{A}\ \phi_q\ d^2x$$

$$-\ d_{nq} \int \phi^\dagger_p \not{A}\ \phi_n d^2x]\ \rangle. \qquad (2.30)$$

To evaluate the right-hand sides of Eqs (2.29) and (2.30), we consider

$$\langle \sum_p \sum_q \bar b_p\ a_q\ Z_{pq} \rangle = \frac{\int D\bar b\ Da \left\{ \sum_p \sum_q \bar b_p a_q Z_{pq} \right\} e^{S_o[a,\bar b] + \Delta S'[a,\bar b]}}{\int Db\ Da\ e^{S_o[a,\bar b]\ +\ \Delta S'[a,\bar b]}}\ ,$$

$$(2.31)$$

where $S_o[a,\bar b] = \sum_r (i\lambda_r - m_o)\bar b_r\ a_r$, and $\Delta S'[a,\bar b] = -e(1-a)\sum_{pq}(\not{A})_{pq}\bar b_p a_q$, with $(\not{A})_{pq} = \int d^2x\ \phi^\dagger_p \not{A}\ \phi_q$. We look upon S_o as the free action,

$\Delta S'$ as the interaction term and $\sum_p \sum_q b_p\, a_q\, Z_{pq}$ as the operator whose connected Green's functions are evaluated in Eq. (2.31). The right-hand side of Eq. (2.31) will then be given by all the *connected* "diagrams" with one insertion of $\sum_p \sum_q \bar{b}_p\, a_q\, Z_{pq}$ with "propagators" $\frac{1}{i\lambda_p - m_o}$ and "vertices" provided by $\Delta S'$. The result is

$$\langle \sum_p \sum_q \bar{b}_p\, a_q\, Z_{pq} \rangle = \sum_p \frac{Z_{pp}}{i\lambda_p - m_o} + e(1-a) \sum_{pn} \frac{Z_{pn}\, (\not{A})_{np}}{(i\lambda_p - m_o)(i\lambda_n - m_o)} + \dots\dots\dots, \qquad (2.32)$$

where the ellipsis stand for terms containing more than one factor of $(\not{A})_{np}$. It will turn out that only the first term on the right-hand side of Eq. (2.32) contributes, and we wish to first concentrate on it and simplify it to a form where it can be easily evaluated and also compared with the results of Alfaro, Urrutia, and Vergara [15] (See also Appendix C).

In Eq. (2.29),

$$Z_{pq} = \sum_n \left[C_{pn} \int \phi_n^\dagger \not{A}\, \phi_q\, d^2x + C_{nq} \int \phi_p^\dagger \not{A}\, \phi_n\, d^2x\right].$$

At *this* stage, we shall introduce an *adhoc* regulator for each of the summations in the first term on the right-hand side of (2.32) so that both the summations are separately well-defined. We replace $Z_{pq} \longrightarrow Z^M_{pq}$, where

$$Z^M_{pq} = \sum_n \left[C_{pn} \int \phi_n^\dagger \not{A}\, \phi_q\, d^2x + C_{nq} \int \phi_p^\dagger \not{A}\, \phi_n\, d^2x\right] e^{-\lambda_p^2/M^2}\, e^{-\lambda_n^2/M^2}. \qquad (2.33)$$

Then the first term on the right-hand side of Eq. (2.32) becomes

$$\sum_p \sum_n \frac{e^{-\lambda_p^2/M^2} e^{-\lambda_n^2/M^2}}{i\lambda_p - m_o} \left[\int d^2y \; \phi_p^\dagger(y) \; i\alpha(y)\gamma_5 \; \phi_n(y) \int \phi_n^\dagger(x) \; \not{A}(x)\phi_p(x) d^2x \right.$$

$$\left. + \int d^2y \; \phi_p^\dagger(y) \; \not{A}(y) \; \phi_n(y) \int d^2x \; \phi_n^\dagger(x) \; i\alpha(x)\gamma_5 \; \phi_p(x) \right] . \qquad (2.34)$$

As all the summations and integrations are convergent, one can use the following identity in the first term:

$$\int d^2x \sum_n \phi_n(y) \; \phi_n^\dagger \; (x) \; e^{-\lambda_n^2/M^2} \; \not{A}(x) \; \phi_p(x)$$

$$= \int d^2x \quad e^{-\not{D}_{ay}^2/M^2} \sum_n \phi_n(y) \; \phi_n^\dagger(x) \; \not{A}(x) \; \phi_p(x)$$

$$= e^{-\not{D}_{ay}^2/M^2} \int d^2x \quad \delta^2(y-x) \; \not{A}(x) \; \phi_p(x)$$

$$= e^{\not{D}_{ay}^2/M^2} \int d^2x \; \delta^2(y-x) \; \not{A}(y) \; \phi_p(x) . \qquad (2.35)$$

Using Eq. (2.35) and a similar identity and then using the cyclic property of the trace, express (2.34) as

$$\mathrm{tr} \left[\int d^2y \; d^2x \left\{ \sum_p \frac{\phi_p(x) \; \phi_p^\dagger(y) \quad e^{-\lambda_p^2/M^2}}{i\lambda_p - m_o} \; i\alpha(y)\gamma_5 \; e^{-\not{D}_{ay}^2/M^2} \delta^2(y-x) \; \not{A}(y) \right. \right.$$

$$\left. \left. + \gamma_5 \sum_p \frac{\phi_p(x) \; \phi_p^\dagger(y)}{i\lambda_p - m_o} \quad e^{-\lambda_p^2/M^2} \not{A}(y) \; e^{-\not{D}_{ay}^2/M^2} \; \delta^2(x-y) \, i\alpha(y) \right\} \right] . \qquad (2.36)$$

Here

$$G_M(x,y) \equiv \sum_p \frac{\phi_p(x)\ \phi_p^\dagger(y)\ e^{-\lambda_p^2/M^2}}{i\lambda_p - m_o} \tag{2.37}$$

is the regularized fermion Green's function.

In a similar manner, we replace $\sum_n C_{nn}$ by $\sum_n C_{nn}\ e^{-\lambda_n^2/M^2}$, and thus obtain

$$\int d^2x\ \alpha(x)\ \langle\ \partial^\mu\ J^A_\mu - 2im_o\ \bar\psi\ \gamma_5\ \psi\ \rangle$$

$$= \lim_{M\to\infty} 2\sum_n C_{nn}\ e^{-\lambda_n^2/M^2} + e(1-a)\lim_{M\to\infty}\sum_n\sum_p \frac{e^{-\lambda_p^2/M^2}\ e^{-\lambda_n^2/M^2}}{i\lambda_p - m_o}\left[C_{pn}(\not A)_{np} + C_{np}(\not A)_{pn}\right] + \ldots\ldots., \tag{2.38}$$

$$= \lim_{M\to\infty} 2\sum_n C_{nn}\ e^{-\lambda_n^2/M^2}$$

$$+ie(1-a)\lim_{M\to\infty} \mathrm{Tr}\left\{\int d^2x d^2y \left[G_M(x,y)\alpha(y)\gamma_5\ e^{-\not D^2_{ay}/M^2}\ \delta^2(x-y)\not A(y) + \gamma_5 G_M(x,y)\ \not A(y)\ e^{-\not D^2_{ay}/M^2}\ \delta^2(x-y)\ \alpha(y)\right]\right\}.$$

$$+\ldots\ldots. \tag{2.39}$$

An identical procedure followed for the vector current leads to

$$\int d^2x\ \alpha(x)\langle\partial^\mu\ J^V_\mu\rangle = -e(1-a)\lim_{M\to\infty}\sum_n\sum_p \frac{e^{-\lambda_p^2/M^2}\ e^{-\lambda_n^2/M^2}}{i\lambda_p - m_o}\left[d_{pn}(\not A)_{np} - d_{np}(\not A)_{pn}\right]$$

$$+ \ldots\ldots\ldots. \tag{2.40}$$

2.5. EVALUATION OF ANOMALIES

In this section, we shall evauluate completely the expressions for the vector and the axial-vector anomalies and obtain the family of anomalies of Eq. (2.1)

A. Axial-Vector Anomaly

First, we shall deal with the term $2 \sum_n C_{nn}\, e^{-\lambda_n^2/M^2}$ in Eq. (2.39). This term is, following the standard procedure [7], $2\int \alpha(x)\ A(x)\ d^2x$, where $A(x) = \lim_{y\to x}\ tr[\gamma_5\ \exp(\not{D}_a^2/M^2)\ \delta^2(x-y)]$.

The answer for A(x) is the same as in the case of the usual choice a = 1 except that A is replaced everywhere by aA. Hence one obtains

$$2 \sum_n C_{nn}\ e^{-\lambda_n^2/M^2} = a\int d^2x\ \alpha(x)\ (e/\pi)\ \varepsilon^{\mu\nu}\ \partial_\mu A_\nu. \qquad (2.41)$$

Next, we proceed to the evaluation of the second term on the right-hand side of Eq. (2.39).

To evaluate this we substitute the momentum representation for $\delta^2(x-y)$, viz.,

$$\delta^2(x-y) = \int \frac{d^2k}{(2\pi)^2}\ e^{ik.(x-y)}, \qquad (2.42)$$

and thus obtain, as an operator

$$e^{-\not{D}_{ay}^2/M^2}\delta^2(x-y) = \int\frac{d^2k\ e^{ik.(x-y)}}{(2\pi)^2}\ e^{-[-i\not{k} + \not{\partial}_y + iea\not{A}(y)]^2/M^2}. \qquad (2.43)$$

We express $G_M(x,y)$ of Eq. (2.37) as

$$G_M(x,y) = e^{-\not{D}_{ax}^2/M^2} \sum_p \frac{\phi_p(x)\ \phi_p^\dagger(y)}{i\lambda_p - m_o}$$
$$\equiv e^{-\not{D}_{ax}^2/M^2}\ G(x,y), \qquad (2.44)$$

and G(x,y) has the expansion in terms of the free Green's function

$$G(x,y) = G_o(x,y) + ea \int G_o(x,z)\, \not{A}(z)\, G_o(z,y)\, d^2z + \ldots\ldots.$$

$$= G_o(x,y) + G_1(x,y) + \ldots\ldots\ldots, \qquad (2.45)$$

where $G_o(x,y)$, the free Green's function, is

$$G_o(x,y) = \int \frac{d^2k}{(2\pi)^2}\, e^{ik.(x-y)}\, \frac{1}{-\not{k}-m_o}\,. \qquad (2.46)$$

Then

$$e^{-\not{D}^2_{ax}/M^2} G_o(x,y) = \int \frac{d^2k}{(2\pi)^2}\, e^{k^2/M^2}\, e^{ik.(x-y)}\, e^{\left\{-\not{D}^2_{ax}/M^2 - \frac{2ik.D_{ax}}{M^2}\right\}} \frac{-1}{\not{k}+m_o}\,. \qquad (2.47)$$

We deal with the contributions to the large parenthesis in Eq. (2.39) from successive terms in Eq. (2.45).

B. Contribution from free Green's function

We substitute in the large parentheses of Eq. (2.39), $e^{-\not{D}^2_{ax}/M^2} G_o(x,y)$ of Eq. (2.47) in place of $G_M\ (x,y)$. We also use Eq. (2.43) and thus obtain the expression

$$ie(1-a)\int d^2x\, d^2y\, \frac{d^2k\, d^2k'}{(2\pi)^4}\, e^{i(k+k').(x-y)}\, e^{k^2/M^2}\, e^{k'^2/M^2}$$

$$\mathrm{Tr}\left\{ e^{\left(\frac{-\not{D}^2_{ax}}{M^2} - \frac{2ik.D_{ax}}{M^2}\right)} \frac{-1}{\not{k}+m_o}\, \alpha(y)\gamma_5\, e^{\left(\frac{-\not{D}^2_{ay}}{M^2} + \frac{2ik'.D_{ay}}{M^2}\right)} \not{A}(y) \right.$$

$$\left. +\ \gamma_5\, e^{\left(\frac{-\not{D}^2_{ax}}{M^2} - \frac{2ik.D_{ax}}{M^2}\right)} \frac{-1}{\not{k}+m_o}\, \not{A}(y)\, e^{\left(\frac{-\not{D}^2_{ay}}{M^2} + \frac{2ik.D_{ay}}{M^2}\right)} \alpha(y) \right\} \qquad (2.48)$$

The terms in (2.48) are dealt with in detail in the Appendix A and it is shown that in the limit $M \longrightarrow \infty$, the contribution comes only from those terms in which $\exp\left\{ \frac{-\not{\phi}^2_{ax}}{M^2} - (\frac{2i}{M^2}) k.D_{ax} \right\}$ is replaced by 1 and the only term in $\exp\left\{ -\frac{\not{\phi}^2_{ay}}{M^2} + (\frac{2i}{M^2}) k'.\ D_{ay} \right\}$ that contributes is $(\frac{2i}{M^2}) k'.D_{ay}$. Carrying out the x-integration, this contribution becomes

$$-2\ e(1-a) \lim_{M\to\infty} \frac{1}{M^2} \left\{ \int d^2y \frac{d^2k}{(2\pi)^2} e^{\frac{2k^2}{M^2}} \left[\frac{\not{k}}{k^2-m_o^2} \alpha(y) \gamma_5 (k.\partial_y) \not{A}(y) \right.\right.$$
$$\left.\left. + \gamma_5 \frac{\not{k}}{k^2-m_o^2} \not{A}(y)\ k.\partial_y\ \alpha(y) \right] \right\} . \qquad (2.49)$$

We rescale $k \to Mk$. Then both terms contribute equally and yield for (2.48),

$$\frac{e}{2\pi} (1-a) \int d^2y\ \alpha(y)\ \varepsilon^{\mu\nu}\ \partial_\mu A_\nu . \qquad (2.50)$$

C. Contributions From G_1, G_2 etc.

These are dealt with in detail in the appendix B and it is shown that these terms are regularized properly and do not contribute as $M \to \infty$.

One, thus, has from Eqs. (2.39), (2.41) and (2.50) the expression for the anomaly

$$\langle \partial_\mu J^A_\mu \rangle - 2im_o \langle \bar{\psi} \gamma_5 \psi \rangle = \frac{e}{2\pi} (1+a)\ \varepsilon^{\mu\nu}\ \partial_\mu A_\nu . \qquad (2.51)$$

D. Vector Anomaly

An identical treatment can be given for the vector anomaly. There is no anologue of Eq. (2.41). Hence the entire contribution is proportional to (1-a). The result is

$$\langle \partial^{\mu} J^{V}_{\mu} \rangle = \frac{e(1-a)}{2\pi} \partial^{\mu} A_{\mu}. \tag{2.52}$$

In Eqs. (2.51) and (2.52), we obtain the family of anomalies of Eq. (2.1).

2.6. CONCLUSIONS AND COMMENTS

In this section we make a few comments on our results and compare our work with some other works addressing the same problem.

In this chapter we have studied the regularization of the path-integral for a two dimensional fermionic system in terms of the eigenfunctions and eigenvalues of the operator $\not{D}_a = \not{\partial} + iea\not{A}$, 'a' being a real, continuous parameter. We derived the W-T identities for local chiral and vector transformations and obtained expressions for $\langle \partial^{\mu} J^{A}_{\mu} \rangle$ and $\langle \partial^{\mu} J^{V}_{\mu} \rangle$. We proposed a straightforward regularization that ultimately lead to the family of anomalies in which the parameter appearing in the family is related to 'a' of $\not{D}_a$.

From equations (2.51) and (2.52) we make some interesting observations. (1) contribution to vector anomaly in Eq. (2.52) arises entirely from the action. There is no Jacobian contribution, since the fermionic measure $D\psi D\bar{\psi}$ is invariant under

the transformations (2.22). One may recall that there was no way at all to derive the vector anomaly in the formalism as shown by Fujikawa [7]. (2) The case a = 0 coincides with the use of the free Dirac operator $\not\partial$ for defining the path-integral and its use is perfectly consistent in our method. The chiral jacobian (see Eq.(2.41)) contributing to $\langle \partial^{\mu} J^{A}_{\mu} \rangle$ is of course zero ($\sum_n C_{nn}\, e^{-\lambda_n^2/M^2}$ in this case is a gauge field independent quantity), and the entire contribution comes from action sources. The use of this operator was disallowed altogether in Fujikawa's method since it led to null results both for the chiral as well as vector anomaly. Our derivation shows, in fact, that identification of anomalies *a priori* as jacobian factors is suspect. (Since this identification is made by formal derivations using unregularized quantities). Anomalies can generally have contributions other than jacobians if we deal with well defined regularized quantities. (Note that "action" terms such as first two terms on the right-hand side of Eq. (2.29) would be formally zero without regularization but they do not vanish even with a finite-mode cut-off). In chapter VI, section (6.5) we give examples which show in an obvious way, how, dealing with well—defined quantities leads to anomaly having contributions other than jacobians. This is true even when the regularization procedure is a gauge invariant one, when use of gauge covariant operators other than $\not D$ is made. (For further elaboration, see sec.(6.5) of chapter VI where we expand on these statements in detail in the light of our very general formulation for anomalies presented there).

References 16 and 17 identify the anomaly, *a priori*, with jacobian factors (an erroneous assumption, as stated above)

which forces them to use non-hermitian operators to derive their results. The construction of such operators is done by exploiting the 2-d γ-matrix structure, making the formalism particular to 2-dimensions. Hermitian combinations $\not{D}^\dagger\not{D}$ & $\not{D}\not{D}^\dagger$ are constructed from the basic parameter dependent operator $\not{D}$, for regularization. In Ref. 17 for example, ψ and $\bar{\psi}$ are expanded in terms of two <u>different</u> bases belonging to $\not{D}^\dagger\not{D}$ and $\not{D}\not{D}^\dagger$ respectively ($\not{D}$ = $i\not{\partial}+\alpha\not{A}+\beta\gamma_5\not{V}$). (This is done to allow for a non-zero value for jacobian for vector anomaly). Our work however, uses a <u>single</u> Hermitian operator whose basis is used to expand both ψ and $\bar{\psi}$ and whose eigenvalues are used for regularization. Moreover, our formalism has a very natural extension in 4-dimensional QED as will be shown later in the thesis. This is unlike in Ref. 16 and 17 where it is not clear how the method can find a straightforward extension in four-dimensions.

We now compare our work with that of Ref. 15. Ref. 15 also uses the operator $\not{D}_a=\not{\partial}+iea\not{A}$ for regularization in the context of massless $(QED)_2$, but in a way quite different in principle, and in detail from our work. In this work, the anomaly is associated with the jacobian $J(\alpha(x))$ defined as

$$J(\alpha) \equiv \frac{\det \not{D}}{\det\,(\not{D} + \not{D}K + L\not{D})} = 1 - \mathrm{Tr}\,[\not{D}^{-1}(\not{D}K + L\not{D})], \qquad (2.53)$$

which is unregularized off hand.

K and L are operators depending on the local parameter $\alpha(x)$ and define the symmetry transformation on ψ and $\bar{\psi}$:

$$\begin{aligned} \psi' &= (1 + K(\alpha))\,\psi, \\ \bar{\psi}' &= \bar{\psi}(1 + L(\alpha)). \end{aligned} \qquad (2.54)$$

The jacobian in Eq. (2.53) is related to the divergence of the current (defined by the change in action for the transformation (2.54)), through

$$0 = \partial^{\mu} \langle J_{\mu}(x) \rangle + \frac{\delta \ln J(\alpha)}{\delta\alpha(x)} \bigg|_{\alpha=0} . \tag{2.55}$$

$J(\alpha)$ of Eq. (2.53) which is ill-defined, is regularized (in an artificial manner) as

$$J_R(\alpha) = 1 - \mathrm{Tr}(\not{D}^{-1}(\not{D}K + L\not{D})R) \equiv 1-T, \tag{2.56}$$

where $R = f(-S/M^2)$ and limit $M^2 \to \infty$ is taken at the end of the calculation. S is an arbitrary hermitian, positive definite operator, and the regularizing function f satisfies the usual Fujikawa conditions [7]. $S = \not{D}_a^2$ is used to derive the family of anomalies.

For the vector and axial-vector transformations in particular, the jacobian factors are (see Eq. (2.56))

$$T^V = -i\ \mathrm{Tr}\ (\not{D}^{-1}\beta\ [\not{D},\ R]), \quad [K = -L = i\beta(x)] \tag{2.57a}$$

$$T^A = i\ \mathrm{Tr}\ [2\alpha(x)\gamma_5 R + \not{D}^{-1}\alpha(x)\gamma_5[\not{D},R]), \ [K = L = i\gamma_5\alpha(x)] \tag{2.57b}$$

$$(R = \exp\ [-\ \not{D}_a^2/M^2]).$$

We shall compare our expressions for the vector and axial-vector anomalies with the ones given above in the work of Ref. 15.

In Sec. (2.4) we found that

$$\int d^2x\ \alpha(x)\ \langle\partial^{\mu}\ J^A_{\mu} - 2im_o\bar{\psi}\gamma_5\psi\rangle^{reg}$$

$$= ie(1-a)tr\Big[\int d^2x\ d^2y\ G_M(x,y)\ \alpha(y)\gamma_5\ e^{-\not{D}^2_{ay}/M^2}\delta^2(x-y)\ \not{A}(y)$$

$$+ \int d^2x\ d^2y\ \gamma_5\ G_M(x,y)\ \not{A}(y)\ e^{-\not{D}^2_{ay}/M^2}\delta^2(x-y)\ \alpha(y)\Big]$$

$$+\ 2\sum_n\ C_{nn}\ e^{-\lambda^2_n/M^2}\ + \ldots\ldots\ldots, \tag{2.58}$$

where the ellipsis denote the terms which do not contribute. The second term on the right-hand side is the usual jacobian term and coincides with the term Tr $[2\alpha\gamma_5 R]$ of Eq. (2.57b) of Ref. 15. We introduce the notation $\alpha(y)\gamma_5\ \delta^2(y-z) = (\alpha\gamma_5)_{ij}$ and $\not{A}(y)\ \delta^2(y-z) = (\not{A})_{ij}$ in an obvious manner, and we write $-i(\not{D}_M^{-1})_{ij} = G_M(y,z)$, $e^{-\not{D}^2_{ay}/M^2}\ \delta^2(y-z) = R_{ij}$. [Here i and j refer to y and z]. Then we replace integrations by summations and thus rewrite the first two terms on the right-hand side of Eq. (2.58) as (with $m_o = 0$)

$$+\ e(1-a)\ Tr\ [\ \not{D}^{-1}_{aM}\ (\alpha\gamma_5)R\ \not{A} + \not{D}^{-1}_{aM}\ \ \not{A}\ R\ (\alpha\gamma_5)]$$

$$= e(1-a)\ Tr\ [\not{D}^{-1}_{aM}\ (\alpha\gamma_5)\ R\ \not{A} + (\alpha\gamma_5)\ \not{D}^{-1}_{aM}\ \not{A}\ R], \tag{2.59}$$

where we have made use of the cyclic property of the trace in the second term in the square brackets (a valid operation since the expressions are regularized). The expression of Eq.(2.59) is to be compared with the expression of Eq. (2.57b) of Ref. 15, obtained after using the identity $[\not{D}, R] = ie(1-a)\ [\not{A}, R]$ of Ref. 15, viz.,

$$e(1-a)\ Tr\Big\{\not{D}^{-1}(\alpha\gamma_5)\ \ [R,\not{A}]\Big\} = e(1-a)\ Tr\Big\{\not{D}^{-1}(\alpha\gamma_5)R\not{A} - \not{D}^{-1}(\alpha\gamma_5)\not{A}R\Big\} \tag{2.60}$$

We note two things. (i) The first term in Eq. (2.59) is

almost identical with the first term in (2.60) except that $\not{\partial}_{aM}^{-1} \longrightarrow \not{\partial}^{-1}$. This has two consequences. This term gives half the contribution as the first term in Eq. (2.60) as a result of this regularization. Secondly, and more importantly, the first term in (2.59) is completely regularized unlike the first term in Eq. (2.60) because (2.60) contains $G(x,y)$ which is unregularized as $x \rightarrow y$ while $G_M(x,y)$ is well-defined as $x \longrightarrow y$. (ii) The second term in Eq. (2.59) is however different from that of Eq. (2.60) and cannot be cast in its form. In fact, the former contributes equal to the first term in Eq. (2.59) while the latter contributes nothing to the anomaly.

Finally, we note that the expression in Eq. (2.59) has arisen from the change in the action and not from the Jacobian, while in Ref. 15, expression (2.60) has arisen from their regularization of the Jacobian.

An analogous discussion applies to the vector case.

We conclude this chapter by a remark. Parameter dependent regulators used in QED for derivation of family structure, have been used to define an anomalous gauge theory like Chiral Schwinger Model as a consistent theory [28-31]. The free parameter acquires a physical significance in this case. (See Ref. 31, for example, for the path-integral treatment of CSM.)

CHAPTER - III

FAMILY OF ANOMALIES IN $(QED)_2$ IN PATH-INTEGRAL FORMULATION

II. DEFINITION OF REGULARIZED CURRENTS.

3.1 INTRODUCTION

In the previous chapter, the family of anomalies in $(QED)_2$ was derived in the path-integral formulation by making use of the gauge-variant operator $\not{D}_a$[13], which was a straightforward generalization of the covariant operator $\not{D}$ of the theory. As was pointed out in that chapter itself, the process followed there is formal upto a stage, in that one proceeds with unregularized quantities upto that stage, obtains an unregularized expression for anomalies and then introduces *an ad-hoc* (but natural) regularization. [This, of course, is the level of rigor on most derivations following Fujikawa's approach]. The purpose of the work in this chapter [14], in part, is to rectify this and have a procedure that is entirely rigorous from the beginning. To this end we adopt an approach somewhat akin to that used by Verstegen [36].

The second motivation for the work in this chapter is the following. In 2-dimensional QED, in the operator formalism, one has the regularized definitions of currents available by point-splitting method [34]. They are

$$J^A_\mu (x,\varepsilon) = \bar{\psi}(x + \varepsilon/2\ ) \gamma_\mu\gamma_5 \exp\left[- iae\int_{x-\varepsilon/2}^{x+\varepsilon/2} A^\nu dy_\nu \right] \psi(x-\varepsilon/2),$$

$$J^V_\mu (x,\varepsilon) = \bar{\psi}\ (x + \varepsilon/2)\ \gamma_\mu\ \exp\left[- iae\int_{x-\varepsilon/2}^{x+\varepsilon/2} A^\nu\ dy_\nu \right]\psi(x-\varepsilon/2). \tag{3.1}$$

This definition, together with the canonical operator formalism, uniquely defines, in a well-defined manner, the Green's functions of J_μ^A and J_μ^V, and as $\varepsilon \to 0$, they lead to the family of anomalies. These definitions naturally contain a free parameter 'a', which defines the family. The following question arises: What is the analogous definition of regularized currents in the path-integral formulation and the quantization procedure enabling one to evaluate their Green's functions in a well-defined manner ? The second purpose of the work in this chapter is to answer this question. The definitions of the regularized currents are

$$
\begin{aligned}
J_\mu^{AM} &= \bar{\psi}(x)\ e^{-\overleftarrow{\not{D}}_a^2/M^2}\ \gamma_\mu\ \gamma_5\ e^{-\not{D}_a^2/M^2}\ \psi(x), \\
J_\mu^{VM} &= \bar{\psi}(x)\ e^{-\overleftarrow{\not{D}}_a^2/M^2}\ \gamma_\mu\ e^{-\not{D}_a^2/M^2}\ \psi(x).
\end{aligned}
\tag{3.2}
$$

Both currents as in Eq. (3.1) are essentially non-local. The quantization procedure is the one formulated in chapter II, viz., you expand ψ and $\bar{\psi}$ in eigenfunctions of $\not{D}_a$ and *define* $D\psi\, D\bar{\psi} = \prod_n da_n \prod d\bar{b}_n$ in terms of the coefficients of expansion a_n and $\bar{b}_n$. The regularizations needed are implicit in Eq. (3.2).

In Sec. (3.2), we give the definitions of the regularized currents. In Sec. (3.3) we find expressions for their expectation values. In sec. (3.4) we evaluate and simplify the expressions for $\langle\partial^\mu J_\mu^A\rangle$ and $\langle\partial^\mu J_\mu^V\rangle$. We find these expressions, not surprisingly, already dealt with thoroughly in Chapter II. We rely on the algebra done in Chapter II and deduce the family of anomalies for currents of Eq.(3.2). The entire procedure is a

regularized one *ab initio.* In section (3.5) we give the conclusions. Though most of the expressions, as far as the algebra involved is concerned, have been dealt with in chapter II, we write this chapter to highlight the definitions of regularized currents given by Eq. (3.2) and to stress the fact that we have a procedure well defined and regularized at all steps. Also, these definitions are the starting point for our subsequent work in the four-dimensional case [19,20,22]. A similar regularization will be used for chiral current in the 2-dimensional non-abelian case, in chapter VII [23].

3.2 PRELIMINARY

In this section we shall establish our notations and give definition of the regularized currents.

A. Notation

We work in two-dimensional QED in Euclidean space. Our notation is the same as in chapter II.

$$\mathcal{L} \;=\; \bar{\psi}(i\not{D}-m_0)\psi \;. \tag{3.3}$$

In particular, for reference, we note that equations (2.2) - (2.8) hold.

B. Definitions of regularized currents

We, now, give the definitions of the regularized currents:

$$J^{AM}_{\mu}(x) \;\equiv\; \bar{\psi}(x)\; e^{-\overleftarrow{\not{D}}^2_a/M^2}\,\gamma_\mu\,\gamma_5\; e^{-\not{D}^2_a/M^2}\,\psi(x), \tag{3.4}$$

$$J^{VM}_{\mu}(x) \equiv \bar{\psi}(x)\; e^{-\overleftarrow{\not{D}}^2_a/M^2}\,\gamma_\mu\; e^{-\not{D}^2_a/M^2}\,\psi(x). \tag{3.5}$$

We observe a number of things. (i) J_μ^{AM} and J_μ^{VM} both depend on a continuous parameter a, introduced solely through the regularizing factors $\exp\{-\not{D}_a^2/M^2\}$. (ii) Both expressions contain derivatives of ψ of an arbitrarily high order and consequently are nonlocal objects. But this is also the case with the point-split regularizations of the vector and axial-vector currents. (iii) These expressions are "regularized" in the sense that $\langle J_\mu^A \rangle$ and $\langle J_\mu^V \rangle$ are well-defined, as seen below.

Similar definitions (with a=1) have been utilized by Verstegen [36] in the four-dimensional context.

3.3 EXPRESSIONS FOR EXPECTATION VALUES OF REGULARIZED CURRENTS

We define the expectation values of currents as

$$\begin{aligned} \langle J_\mu^{AM} \rangle &\equiv \frac{1}{W[A]} \int D\bar{\psi}\, D\psi \;\; J_\mu^{AM} \;\; e^S, \\ \langle J_\mu^{VM} \rangle &\equiv \frac{1}{W[A]} \int D\bar{\psi}\, D\psi \;\; J_\mu^{VM} \;\; e^S, \end{aligned} \tag{3.6}$$

and evaluate them below. First, let us evaluate J_μ^{AM}.

We substitute the expansions of Eq. (2.4) in the expression for J_μ^{AM} and use Eq. (2.3) and obtain

$$J_\mu^{AM}(x) = \sum_p \sum_q \bar{b}_p \, a_q \, X_{pq\mu}^A(x), \tag{3.7}$$

with

$$X_{pq\mu}^A(x) \equiv \phi_p^\dagger \, \gamma_\mu \gamma_5 \, \phi_q \;\; e^{-\lambda_p^2/M^2} \;\; e^{-\lambda_q^2/M^2}. \tag{3.8}$$

To evaluate $\langle J_\mu^{AM} \rangle$ completely, we shall proceed as in chapter II. As explained there [See Eq. (2.32)],

$$\langle \sum_p \sum_q \bar{b}_p\, a_q\, z_{pq} \rangle = \sum_p \frac{z_{pp}}{i\lambda_p - m_o} + e(1-a) \sum_p \sum_q \frac{z_{pq}\, (\not{A})_{qp}}{(i\lambda_p - m_o)(i\lambda_q - m_o)}$$

$$+ [e(1-a)]^2 \sum_p \sum_q \sum_m \frac{z_{pq}\, (\not{A})_{qm}\, (\not{A})_{mp}}{(i\lambda_p - m_o)(i\lambda_q - m_o)(i\lambda_m - m_o)}$$

$$+ \ldots\ldots\ldots\ldots \quad . \tag{3.9}$$

We let $z_{pq} = X^A_{pq\mu}$ to obtain

$$\langle J_\mu^{AM} \rangle = \sum_p \frac{X^A_{pp\mu}}{i\lambda_p - m_o} + e(1-a) \sum_p \sum_q \frac{X^A_{pq\mu}\, (\not{A})_{qp}}{(i\lambda_p - m_o)(i\lambda_q - m_o)}$$

$$+ \ldots\ldots\ldots\ldots \quad . \tag{3.10}$$

Eq. (3.10) gives a completely regularized expression for the axial-vector current, $\langle J_\mu^{AM} \rangle$. (That this expression is regularized can be seen in the same manner as the terms coming from higher order Green's functions in chapter II were shown to be regularized).

In a similar manner, one obtains regularized expression for the vector current, $\langle J_\mu^{VM} \rangle$, which is identical with the right hand side of (3.10) except that

$$X^A_{pq\mu} \longrightarrow X^V_{pq\mu} = \phi_p^\dagger\, \gamma_\mu\, \phi_q\, e^{-\lambda_p^2/M^2}\, e^{-\lambda_q^2/M^2} .$$

3.4 DIVERGENCES OF REGULARIZED CURRENTS

We obtain the expression for the divergence of $\langle J_\mu^{AM}\rangle$ by using

$$\partial^\mu X^A_{pq\mu} = -\left[\lambda_p\, \phi_p^\dagger\, \gamma_5 \phi_q + \lambda_q\, \phi_p^\dagger\, \gamma_5\phi_q\right] e^{-\lambda_p^2/M^2}\, e^{-\lambda_q^2/M^2} \tag{3.11}$$

$$= i\left[(i\lambda_p - m_o)\phi_p^\dagger\, \gamma_5\phi_q + (i\lambda_q - m_o)\phi_p^\dagger\, \gamma_5\phi_q\right] e^{-\lambda_p^2/M^2}\, e^{-\lambda_q^2/M^2}$$

$$+\ 2im_o\, \phi_p^\dagger\, \gamma_5\phi_q\, e^{-\lambda_p^2/M^2}\, e^{-\lambda_q^2/M^2}\,. \tag{3.12}$$

The second term on the right-hand side of Eq. (3.12) when substituted in the successive terms of the right-hand side of Eq. (3.10) gives rise to the series

$$2im_o\left[\sum_p \frac{\phi_p^\dagger\, \gamma_5\phi_p\, e^{-2\lambda_p^2/M^2}}{i\lambda_p - m_o} + e(1-a)\sum_p\sum_q \frac{\phi_p^\dagger\, \gamma_5\phi_q\, (\not{A})_{qp}\, e^{-\lambda_p^2/M^2}\, e^{-\lambda_q^2/M^2}}{(i\lambda_p - m_o)\,(i\lambda_q - m_o)} + \cdots\cdots\cdots\right], \tag{3.13}$$

which is seen with the help of Eq. (3.9) to be identical with $\langle\, 2im_o\, \bar\psi\, e^{-\not{D}_a^2/M^2}\gamma_5\, e^{-\not{D}_a^2/M^2}\psi\,\rangle$, and we define this to be $2im_o\,\langle J_5^M\rangle$.

The first term on the right-hand side of Eq. (3.12), when substituted in Eq. (3.10) yields

$$2i\sum_m \phi_m^\dagger\, \gamma_5\phi_m\, e^{-2\lambda_m^2/M^2} + ie(1-a)\left[\sum_p\sum_n \frac{\phi_p^\dagger\, \gamma_5\phi_n\, (\not{A})_{np}}{i\lambda_p - m_o}\right.$$

$$\left. + \sum_p\sum_n \frac{\phi_p^\dagger\, \gamma_5\phi_n\, (\not{A})_{np}}{i\lambda_n - m_o}\right] + \cdots\cdots \tag{3.14}$$

$$= 2i \sum_m \phi_m^\dagger \gamma_5 \phi_m \, e^{-2\lambda_m^2/M^2} + ie(1-a) \sum_p \sum_n \frac{1}{(i\lambda_p - m_o)} \Big\{ \phi_p^\dagger \gamma_5 \phi_n (\not{A})_{np}$$

$$+ \phi_n^\dagger \gamma_5 \phi_p (\not{A})_{pn} \Big\} + \ldots\ldots. \qquad (3.15)$$

Thus one has, from Eqs. (3.12) and (3.15),

$$\langle \partial^\mu J_\mu^{AM} - 2im_o J_5^M \rangle = 2i \sum_m \phi_m^\dagger \gamma_5 \phi_m \, e^{-2\lambda_m^2/M^2}$$

$$+ ie(1-a) \sum_p \sum_n \frac{1}{(i\lambda_p - m_o)} \Big\{ \phi_p^\dagger \gamma_5 \phi_n (\not{A})_{np} + \phi_n^\dagger \gamma_5 \phi_p (\not{A})_{pn} \Big\}$$

$$+ \ldots\ldots\ldots \, . \qquad (3.16)$$

To compare this result with the corresponding result in chapter II, we multiply Eq. (3.16) by $\alpha(x)$ and integrate over x and thus obtain,

$$\int d^2x \, \alpha(x) \, \langle \partial^\mu J_\mu^{AM} - 2im_o J_5^M \rangle$$

$$= 2 \sum_n C_{nn} \, e^{-2\lambda_n^2/M^2} + e(1-a) \sum_p \sum_n \frac{e^{-\lambda_p^2/M^2}}{(i\lambda_p - m_o)} \, e^{-\lambda_n^2/M^2}$$

$$\times \Big[\int \phi_p^\dagger(x) \gamma_5 \, i\alpha(x) \, \phi_n(x) \, (\not{A})_{np} \, d^2x$$

$$+ \int \phi_n^\dagger(x) \, i\alpha(x) \gamma_5 \, \phi_p(x) \, (\not{A})_{pn} \, d^2x \Big] + \ldots\ldots \qquad (3.17)$$

We compare Eq. (3.17) with Eq. (2.38) of chapter II and note that apart from the *ad-hoc* regulator $e^{-\lambda_n^2/M^2}$ in the first term on the right-hand side of Eq. (2.38) being replaced by

$e^{-2\lambda_n^2/M^2}$, the two expressions are identical. The change in the regulator for the jacobian term leads to no difference as M —> ∞.

In an identical manner, we find the identity for the vector current, and it is verified to be the same as Eq. (2.40) of chapter II. The rest of the details of the calculation are as in chapter II and lead to the result

$$\lim_{M\to\infty} \langle \partial^{\mu} J_{\mu}^{AM} - 2im_o J_5^M \rangle = \frac{e}{2\pi} (1+a) \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu} , \tag{3.18a}$$

and

$$\lim_{M\to\infty} \langle \partial^{\mu} J_{\mu}^{VM} \rangle = \frac{e(1-a)}{2\pi} \partial^{\mu} A_{\mu} , \tag{3.18b}$$

the familiar system of anomalies.

3.5 CONCLUSION

In this chapter, we gave definitions of regularized currents which could be taken in the path-integral formulation, as analogues of currents regularized by point-splitting techniques, in the canonical operator formalism. The definition of these regularized currents involved expansion of the fermion fields in terms of basis of the eigenfunctions of the herrmitian operator $\not{D}_a$ and defining the regularization in terms of eigenvalues of it. With this recipe for defining the currents and the quantization procedure, Eqs. (3.18) were derived in a sequence which was <u>well defined at each step</u>. The anomaly terms were obtained in a regularized form, and no *adhoc* regularization was given at an

intermediate stage to define these quantities. The algebra involved in evaluation of $\langle\partial^{\mu}J_{\mu}^{AM}\rangle$ and $\langle\partial^{\mu}J_{\mu}^{VM}\rangle$ was however, effectively the same as that given in chapter II. The procedure presented here gives a rigorous justification of the formal procedure followed in chapter II.

CHAPTER - IV

FAMILY OF ANOMALIES IN FOUR DIMENSIONS IN PATH-INTEGRAL FORMULATION

4.1 INTRODUCTION

The existence of a family of anomalies in $(QED)_4$ has been known for a long time [2,6], such a family being the essence of the ABJ triangle anomaly [3]. It is well known that this chiral anomaly owes its existence to the fact that gauge invariance and chiral symmetry (upto mass terms), both characterizing the classical action, do not hold simultaneously when the system is quantized. What happens, can be seen in the following way.

Consider the three-point function of electrodynamics, $T_{\mu\nu\lambda}$[18]:

$$T_{\mu\nu\lambda}\ (k_1, k_2, q) = i \int d^4x_1\ d^4x_2 \langle 0|[T(J^V_\mu(x_1)\ J^V_\nu(x_2) J^A_\lambda(0)]|0\rangle$$

$$\times\ e^{ik_1.x_1 + ik_2.x_2} \quad . \qquad (4.1)$$

In the usual Feynman diagrammatic treatment of perturbation theory, $T_{\mu\nu\lambda}$ is ambiguous because it has, in one loop order, a (superficially) linearly divergent diagram. In the usual perturbation theory this ambiguity arises as a result of the possibility of choosing the loop momentum differently. This arbitrariness leads to a free parameter β[18] on which the definition of $T_{\mu\nu\lambda}$ depends. In terms of this $T_{\mu\nu\lambda}(\beta)$, one can express the family of anomalies as [18]

$$q^{\lambda}\, T_{\mu\nu\lambda}(\beta) = 2m\, T_{\mu\nu}(0) - \frac{1-\beta}{4\pi^2}\, \varepsilon_{\mu\nu\sigma\rho}\, k_1^{\sigma}\, k_2^{\rho} \tag{4.2a}$$

and

$$k_1^{\ \mu}\, T_{\mu\nu\lambda}(\beta) = \frac{1+\beta}{8\pi^2}\, \varepsilon_{\nu\lambda\sigma\rho}\, k_1^{\ \sigma}\, k_2^{\ \rho}\ , \tag{4.2b}$$

where

$$T_{\mu\nu}\,(k_1, k_2, q, \beta) = i \int d^4x_1\, d^4x_2 \,\langle\, 0|T\, [J^V_{\mu}\,(x_1) J^V_{\nu}(x_2)\bar{\psi}(0)\gamma_5\psi(0)]\,|0\rangle \times e^{ik_1 . x_1 + ik_2 . x_2} \ . \tag{4.3}$$

Imposing gauge invariance by fixing $\beta = -1$ gives in Eq. (4.2a), the familiar chiral anomaly. Parameter dependent point-splitting regularizations for the axial-vector current [34] were also used to show this conflict between gauge invariance and chiral symmetry, in the canonical operator formalism.

Though the chiral anomaly was derived in the path-integral formalism by Fujikawa [7] the underlying family structure was not established. As already explained in the two-dimensional context, in the formalism as presented by him it was not possible to establish this. The basic question is, what is the analogue in the path-integral formulation, of the momentum routing ambiguity in the usual Feynman diagrammatic treatment of anomalies, and how this ambiguity is to be mathematically formulated so as to make explicit the family structure of Eqns. (4.2) ? Questions related to the derivation of the family structure using parameter dependent regularizations in path-integral formulation in four-dimensions were raised [37], but no explicit relations were derived.

In this chapter we derive family of anomalies in four-dimensions the path-integral framework. This is done by using a basis consisting of the eigenfunctions of operator $\not{D}_a \equiv \not{\partial} + iea\not{A}$ to define the path integral, and its eigenvalues to define the regularization, in a way analogous to the two-dimensional case [19]. In chapter III we had used definition of regularized currents to derive family of anomalies, and this treatment was rigorous and well-defined *ab-initio*. In view of this, our treatment here follows the treatment of chapter III, in that, we begin with definition of regularized currents to derive the family.

The family equations that we establish are in terms of a regularized axial-vector current J_μ^{AM} defined as

$$J_\mu^{AM} = \bar{\psi}(x) \quad \exp\left[\frac{-\not{D}_a^{\,2}}{M^2}\right] \gamma_\mu \, \gamma_5 \, \exp\left[\frac{-\not{D}_a^{\,2}}{M^2}\right] \psi(x) . \tag{4.4}$$

(This expression is "regularized" in the sense that W_ν [A] given below by Eq. (4.5) is well defined).

We further define:

$$W_\nu \,[A] = \lim_{M\to\infty} \frac{\int D\bar{\psi}\; D\psi \quad J_\nu^{AM}(x) \; e^S}{\int D\bar{\psi}\; D\psi \quad e^S} \tag{4.5}$$

$$= \lim_{M\to\infty} \; \langle \, J_\nu^{AM} (x) \, \rangle \, .$$

In terms of this $W_\nu[A]$, we establish analogues of Eqs. (1.2); viz.,

$$\partial^\nu W_\nu \,[A] = \frac{ie^2(1-\beta)}{16\pi^2} \; F_{\mu\nu} \; \tilde{F}^{\mu\nu} + 2im_o \; W_p[A] \tag{4.6a}$$

and

$$\partial_Y^{\mu} \frac{W_\nu\ [A(x)]}{\delta A_\mu(Y)} = \frac{ie^2(1+\beta)}{16\pi^2} \varepsilon_{\nu\mu\rho\sigma} F^{\mu\rho}(x)\partial_x^{\sigma} \delta^4(x-y), \tag{4.6b}$$

where

$$W_P[A] = \lim_{M\to\infty} \langle \bar{\psi}\, e^{-\bar{\not{D}}_a^2/M^2} \gamma_5\, e^{-\not{D}_a^2/M^2} \psi \rangle. \tag{4.6c}$$

(We shall find that β is related to the parameter 'a' by

$$\beta = - a^2). \tag{4.7}$$

The family equations given by Eqs. (4.6a) and (4.6b) deserve an explanation. In the usual Feynman diagrammatic perturbative treatment of anomalies, the family of anomalies of Eqs. (4.2), refers to a <u>single</u> Green's function, $T_{\mu\nu\lambda}$. The analogue of equations describing family of anomalies in the path-integral formulation must also be expressed in terms of a <u>single</u> regularized quantity. The chiral anomaly equation is expressed in terms of $\langle J_\nu^{AM}[A]\rangle \equiv W_\nu^M$ whose second order derivative with respect to A gives the two point function of the axial-vector current J_ν^A . This Green's function is related to $T_{\alpha\beta\nu}$ of perturbation theory. The vector part of the family of anomaly equations, viz. the analogue of Eq. (4.2b) must be expressed in terms of this <u>same</u> quantity.

$$\frac{\delta^2 W_\nu^M[A]}{\delta A_\alpha\ \delta A_\beta} \quad \longleftrightarrow \quad T_{\alpha\beta\nu}$$

Thus, in the path-integral formulation, the left-hand side of the analogue of Eq. (4.2b) must be expressed in terms of $\partial_Y^{\mu}\frac{\delta\ W_\nu^M\ [A]}{\delta A_\mu(Y)}$, whose first-order derivative with respect to A_β is related to the $k_1{}^{\mu}\ T_{\mu\beta\nu}$ of Eq. (4.2b).

The quantity $\partial^{\mu} \frac{\delta W_{\nu}^{M}[A]}{\delta A_{\mu}}$ appears naturally in the change in the gauge transformation of $W_{\nu}^{M}[A]$. Of course, our regularization of J_{μ}^{A} is not gauge-invariant and hence $\partial^{\mu} \frac{\delta W_{\nu}}{\delta A_{\mu}}$ need not vanish and indeed will provide the nonvanishing vector anomaly.

We could also think of giving a family structure using regularized definitions of <u>both</u> vector and axial-vector currents in a symmetric fashion. This is done in appendix G. Ref. 33 have also derived the family structure in four dimensions using the operator $\not{D}_a$. However, the family structure they look for is quite different from ours and is derived by a method which differs conceptually and otherwise from our treatment.

In section (4.2) we define the notations. In section (4.3) we give the derivation of the chiral anomaly equation (4.6a) involving the divergence of the axial-vector current. We indicate only the main steps since this derivation is very similar to that already given in chapter III for the two-dimensional case. Section (4.4) is devoted to the evaluation of the anomaly terms in the chiral anomaly equation. In section (4.5) the vector anomaly equation (4.6b) is derived. It is hence shown that the parameter β in eqns. (4.6) is related to the parameter 'a' appearing in $\not{D}_a$ by $\beta = -a^2$. Thus arbitrariness appearing in the family equations, is seen in the path-integral framework, to be related to the arbitrariness in the choice of basis functions in which to expand the fermion fields and define the path-integral. The contribution to anomaly comes from both the measure (jacobian) and the action. We also make a few comments on our results, in this

section. Mathematical details of calculations are given in appendices D, E and F.

4.2 PRELIMINARY

We consider the Euclidean Lagrangian for four dimensional QED

$$\mathcal{L} = \bar{\psi} (i \not{D} - m_o) \psi , \tag{4.8}$$

where $\not{D} = \gamma^\mu D_\mu = \gamma^\mu (\partial_\mu + ieA_\mu)$, and A_μ is a real abelian external field. We shall mainly use notations of Ref. 7. The γ-matrices are those of Bjorken and Drell [38]: $\gamma^k (k=1,2,3)$ and $\gamma^4 = i\gamma^o$ are all anti-hermitian. γ_5 defined by $\gamma_5 = i\gamma^o\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma^4$ is hermitian. We define $\varepsilon_{1234} = \varepsilon^{1234} = 1$. The metric in Euclidean space is $g_{\mu\nu} = \mathrm{diag}\,(-1,-1,-1,-1)$.

As in chapter II [13], we shall define the path-integral measure by first expanding ψ and $\bar{\psi}$ in terms of the eigenfunctions of a hermitian operator $\not{D}_a = \not{\partial} + iea\not{A}$, where a is a real continuous parameter. We let

$$\begin{aligned} \not{D}_a \, \phi_n (x) &= \lambda_n \phi_n (x), \\ \phi_n^\dagger (x) \overleftarrow{\not{D}}_a &= \lambda_n \phi_n^\dagger (x), \end{aligned} \tag{4.9}$$

where λ_n are the real eigenvalues. We expand

$$\begin{aligned} \psi(x) &= \sum_n a_n \phi_n (x), \\ \bar{\psi}(x) &= \sum_n \phi_n^\dagger (x) \bar{b}_n \end{aligned} \tag{4.10}$$

and define the measure in the path-integral as

$$D\bar{\psi}\, D\psi \;=\; \prod_m d\bar{b}_m \prod_n da_n \;. \tag{4.11}$$

The path-integral W [A] is

$$W\,[A] \;=\; \int D\psi\, D\bar{\psi} \;\; e^S \;\equiv\; \int \prod_m d\bar{b}_m \prod_n da_n \; e^S \;. \tag{4.12}$$

where S is expressed in terms of a_n and $\bar{b}_m$. It reads,

$$S \;=\; \sum_p \sum_q \; \bar{b}_p \; a_q \;\; \xi_{pq} \;, \tag{4.13}$$

where

$$\xi_{pq} \;\equiv\; (i\lambda_q - m_o)\,\delta_{pq} - e(1-a) \int d^4x \; \phi_p^\dagger \, \not{A} \; \phi_q \;. \tag{4.14}$$

4.3 DIVERGENCE OF THE AXIAL-VECTOR CURRENT

In this section, we shall give in a somewhat brief fashion the derivation of the axial-vector anomaly equation (4.6a). We rely considerably on the procedure already utilized in chapter III and give only the main steps, the algebra here being very similar to the two-dimensional case.

We define the regularized generating functional with one insertion of the axial current J_ν^{AM} (x) (defined in Eq. (4.4)) by

$$W_\nu^M\,[A] \;=\; \frac{1}{W[A]} \int D\psi\, D\bar{\psi} \;\; J_\nu^{AM}(x) \;\, e^S \tag{4.15}$$

Using equations (4.9) and (4.10) we find that

$$J_\nu^{AM}(x) \;=\; \sum_p \sum_q \; \bar{b}_p \; a_q \; X_{pq\nu}^A \;(x) \tag{4.16}$$

with

$$X_{pq\nu}^A(x) \;=\; \phi_p^\dagger(x) \;\; \gamma_\nu \gamma_5 \phi_q(x) \;\; \exp\left[\frac{-\lambda_p^2 - \lambda_q^2}{M^2} \right] \tag{4.17}$$

$W_\nu^M[A]$ can be evaluated by performing the Grassmann integration over a_m and $\bar{b}_n$ (for the result, see equation (3.10)).

We obtain the divergence of $W_\nu^M[A]$, proceeding in much the same way as in section (3.4) of chapter III. Using Eq. (3.12) and defining

$$W_P^M [A] = \langle \bar{\psi}(x)\, e^{-\bar{\not{D}}_a^2/M^2} \gamma_5\, e^{-\not{D}_a^2/M^2} \psi(x) \rangle$$

$$= \sum_p \frac{\phi_p^\dagger \gamma_5 \phi_p\, e^{-2\lambda_p^2/M^2}}{i\lambda_p - m_o} + e(1-a) \sum_p \sum_q \frac{\phi_p^\dagger \gamma_5 \phi_q (\not{A})_{qp} \exp\left[\frac{-\lambda_p^2 - \lambda_q^2}{M^2}\right]}{(i\,\lambda_p - m_o)(i\,\lambda_q - m_o)}$$

$$+ [e(1-a)]^2 \sum_p \sum_q \sum_m \frac{\phi_p^\dagger \gamma_5 \phi_q (\not{A})_{qm} (\not{A})_{mp} \exp\left[\frac{-\lambda_p^2 - \lambda_q^2}{M^2}\right]}{(i\,\lambda_p - m_o)(i\,\lambda_q - m_o)(i\,\lambda_m - m_o)}$$

$$+ \ldots\ldots\ldots \qquad (4.18)$$

we obtain

$$\partial^\nu W_\nu^M [A] = 2im_o W_P^M [A] + 2i \sum_p \phi_p^\dagger \gamma_5 \phi_p\, e^{-2\lambda_p^2/M^2}$$

$$+ ie(1-a)\left\{ \sum_{pm} \phi_p^\dagger \gamma_5 \phi_m (\not{A})_{mp} \exp\left[\frac{-\lambda_p^2 - \lambda_m^2}{M^2}\right] \left[\frac{1}{(i\lambda_m - m_o)} + \frac{1}{(i\lambda_p - m_o)}\right]\right\}$$

$$+ i[e(1-a)]^2 \left\{\sum_{pms} \phi_p^\dagger \gamma_5 \phi_m (\not{A})_{ms} (\not{A})_{sp} \exp\left[\frac{-\lambda_p^2 - \lambda_q^2}{M^2}\right]\right.$$

$$\left. \times \left[\frac{1}{(i\lambda_s - m_o)(i\lambda_m - m_o)} + \frac{1}{(i\lambda_s - m_o)(i\lambda_p - m_o)}\right]\right\}$$

$$+ \ldots\ldots\ldots \qquad (4.19)$$

The higher order terms in Eq.(4.19) do not contribute to

the anomaly equations. This can be shown in much the same way as was shown for the corresponding higher order terms in the two dimensional case. [See Eq. (2.32) of chapter II and Appendix C]. It should also be remarked that unlike the two dimensional case, the fourth term on the right hand side of Eq. (4.19) contributes in the four dimensional case.

In the next section, we sketch briefly the evaluation of the last three terms on the right hand side of Eq. (4.19). The evaluation is somewhat complicated here as compared to the two dimensional case. While the third term on the right hand side of Eq. (4.19) is similar to the two dimensional case, the contribution to it [See Eq. (4.21) below] comes not only from the free Green's function, but also from the next term G_1 in the expansion of the Green's function; and the fourth term is entirely new here, but its evaluation is, however, very similar to that of the G_1 term just mentioned.

4.4 EVALUATION OF THE CHIRAL ANOMALY

In this section, we shall outline the evaluation of the anomaly terms on the right-hand side of the chiral anomaly equation (4.19).

Consider first $\sum_p \phi_p^\dagger \gamma_5 \phi_p \, e^{-2\lambda_p^2/M^2}$. The evaluation of this term proceeds as was done by Fujikawa [7] except for two differences (i) M^2 in his case is replaced by $M^2/2$ in our case (ii) ϕ_p and λ_p are respectively the eigenfunctions and eigenvalues of $\not{D}_a$ rather than $\not{D}$. As the result for $M \to \infty$, for this term under consideration, does not depend on M, the first

difference is immaterial. The second change replaces A by aA everywhere. Thus following the result in Ref. 7, one obtains.

$$\lim_{M\to\infty} 2i \sum_p \phi_p^\dagger \gamma_5 \phi_p\, e^{-2\lambda_p^2/M^2} = \frac{ie^2a^2}{4\pi^2} \epsilon^{\mu\nu\lambda\sigma}\, \partial_\mu A_\nu \partial_\lambda A_\sigma$$
$$= \frac{ie^2a^2}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{4.20}$$

Now we shall consider the third term on the right hand side of Eq. (4.19). Calling this term $A_2(x)$, we can cast it in a form more useful for evaluation, as was done in Sec. (2.4) of chapter II. The result [see Eq. (2.39) of chapter II]is

$$\int d^4x\, \alpha(x)\, A_2(x)$$

$$= ie(1-a)\,\mathrm{Tr} \left\{\int d^4x\, d^4y\, [G_M(x,y)\, \alpha(y)\gamma_5 \exp\left(- \frac{\not{D}_{ay}^2}{M^2}\right) \delta^4(x-y)\, \not{A}(y)\right.$$

$$\left. + \gamma_5\, G_M(x,y)\, \not{A}(y) \exp\left(- \frac{\not{D}_{ay}^2}{M^2}\right) \delta^4(x-y)\, \alpha(y), \right. \tag{4.21}$$

where

$$G_M(x,y) = \sum_p \frac{\phi_p(x)\, \phi_p^\dagger(y) \exp\left(- \frac{\lambda_p^2}{M^2}\right)}{i\lambda_p - m_o} \tag{4.22}$$

$$\equiv \exp\left(- \frac{\not{D}_{ax}^2}{M^2}\right) G(x,y) \tag{4.23}$$

The contribution to the right-hand side of Eq. (4.21) comes from the first two terms in the expansion of G (x,y) :

$$G(x,y) = G_o(x,y) + ea \int d^4z\, G_o(x,z)\, \not{A}(z)\, G_o(z,y) + \ldots..$$

$$= G_o + G_1 + \ldots.. \tag{4.24}$$

$$\lim_{M\to\infty} \int d^4x\ \alpha(x)\ A_{21}(x) = \frac{ie^2 a\ (1-a)}{16\pi^2} \int d^4x\ \alpha(x)\ F_{\mu\nu}\ \tilde{F}^{\mu\nu}. \qquad (4.29)$$

Eqs. (4.28) and (4.29) lead to

$$\lim_{M\to\infty} A_2(x) = \frac{ie^2 a\ (1-a)}{8\pi^2}\ F_{\mu\nu}\ \tilde{F}^{\mu\nu}. \qquad (4.30)$$

In the limit $M \to \infty$, the higher terms in the expansion of $G(x,y)$ of Eq. (4.24) do not contribute. This can be shown in much the same way as in Appendix B.

The evaluation of the fourth term $A_3(x)$ on the right-hand side of Eq. (4.19) proceeds much the same way as the contribution of G_1 term and this too is dealt with in Appendix E. The result is

$$\lim_{M\to\infty} A_3(x) = \frac{ie^2(1-a)^2}{16\pi^2}\ F_{\mu\nu}(x)\ \tilde{F}^{\mu\nu}(x). \qquad (4.31)$$

The higher terms in Eq. (4.19) indicated by ellipsis do not contribute as $M \to \infty$. The proof proceeds as in the two dimensional case. See Appendix C.

The net result is expressed by the anomaly equation

$$\partial^\nu W_\nu\ [A] = 2im_o\ W_p\ [A] + \frac{i\ e^2}{16\pi^2}\ [\ 1+ a^2\ ]\ F_{\mu\nu}\ \tilde{F}^{\mu\nu}. \qquad (4.32)$$

Here $W_\nu = \lim_{M\to\infty} W_\nu^M$ and $W_p = \lim_{M\to\infty} W_p^M$.

This can be cast in the form

$$\partial^\nu W_\nu\ [A] = 2im_o\ W_p\ [A] + \frac{i\ e^2}{16\pi^2}\ (1 - \beta)\ F_{\mu\nu}\ \tilde{F}^{\mu\nu} \qquad (4.33)$$

Where one finds that the parameter β is related to the parameter 'a' by

$$\beta = - a^2 . \tag{4.34}$$

4.5 VECTOR ANOMALY EQUATION

In this section we establish the vector anomaly equation, viz., equation (4.6b). We proceed in the following way.

Consider the change of variables

$$\begin{aligned} \psi(x) &= e^{i\alpha(x)} \psi'(x), \\ \bar{\psi}(x) &= \bar{\psi}'(x) e^{-i\alpha(x)} \end{aligned} \tag{4.35}$$

and let

$$A_\mu(x) \equiv A'_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \tag{4.36}$$

in the expression for $W_\nu^M [A]$, viz.,

$$W_\nu^M [A] = \frac{1}{W[A]} \int D\psi D\bar{\psi} \bar{\psi}(x) \exp\left(- \frac{\overleftarrow{\not{D}}^2_{ax}}{M^2}\right)\gamma_\nu\gamma_5 \exp\left(- \frac{\not{D}^2_{ax}}{M^2}\right) \psi(x) e^S. \tag{4.37}$$

Noting

$$D_{a\mu}[A] e^{i\alpha(x)} = e^{i\alpha(x)} \left\{ D_{a\mu}[A'] + i(1-a)\partial_\mu\alpha(x)\right\} \tag{4.38}$$

and thus

$$f(D_{a\mu}[A]) e^{i\alpha(x)} = e^{i\alpha(x)} f(D_{a\mu}[A'] + i(1-a)\partial_\mu\alpha) \tag{4.39}$$

and also

$$e^{-i\alpha(x)} f(\overleftarrow{D}_{a\mu}[A]) = f (\overleftarrow{D}_{a\mu}[A'] + i(1-a)\partial_\mu\alpha) e^{-i\alpha(x)} . \tag{4.40}$$

$W_\nu^M [A]$ is expressed also as

$$W_\nu{}^M\,[A] = \frac{1}{W\,[A']} \int D\psi'\; D\bar{\psi}'\; e^{S[\psi',\bar{\psi}',A']}$$

$$\times\; \bar{\psi}'(x)\; \exp\left\{-\,(\bar{\not{D}}_a[A'] + i(1-a)\not{\partial}\alpha)^2/M^2\right\}\, \gamma_\nu\gamma_5$$

$$\times\; \exp\left\{-\,(\not{D}_a[A'] + i(1-a)\not{\partial}\alpha)^2/M^2\right\}\, \psi'(x)$$

$$\equiv\; W'^{\,M}_\nu\,[A'] \tag{4.41}$$

where we have used the invariance of the measure and of the actional form under transformations of Eqs. (4.35) and (4.36), and gauge invariance of W [A]. We write

$$W^M_\nu\,[A] = W_\nu{}^M\,[A'] + W^M_{\nu 1}\,[A] + O(\alpha^2) \tag{4.42}$$

where the second term on the right hand side of Eq.(4.42), $W^M_{\nu 1}[A]$, is linear in α.

We write

$$W_\nu{}^M\,[A'_\mu] = W_\nu{}^M\,[A_\mu + \frac{1}{e}\,\partial_\mu\,\alpha\,]$$

$$= W_\nu{}^M\,[A] + \frac{1}{e}\int d^4y\;\partial_\mu\;\alpha(y)\;\frac{\delta\; W_\nu^M\,[A]}{\delta\;A_\mu(y)} + O(\alpha^2). \tag{4.43}$$

Comparing Eqs. (4.42) and (4.43), we obtain

$$\frac{1}{e}\int d^4y\quad \alpha(y)\;\partial^\mu\;\frac{\delta\; W_\nu{}^M[A]}{\delta\; A_\mu(y)} = W^M_{\nu 1}\,[A]. \tag{4.44}$$

So that Eq. (4.42) yields

$$W^M_{\nu 1}\,[A] = W^M_\nu\,[A] - W_\nu{}^M\,[A'] + O(\alpha^2) \tag{4.45}$$

$$= W'^{\,M}_\nu\,[A'] - W_\nu{}^M\,[A'] + O(\alpha^2) \tag{4.46}$$

The right-hand side of Eq. (4.46) is already of $O(\alpha)$ as seen from the form of the right hand side of Eq. (4.41). Hence, we may replace A′ by A and thus obtain

$$W_{\nu 1}^{M}[A] = W_{\nu}^{\prime M}[A] - W_{\nu}^{M}[A] + O(\alpha^2) \tag{4.47}$$

To evaluate the right-hand side of Eq. (4.47) we need expansions of the exponential operators to $O(\alpha)$:

$$\exp\left\{-(\not{p}_a[A] + i(1-a)\not{\partial}\alpha)^2/M^2\right\} = \exp\left\{-\frac{\not{p}_a^2}{M^2} - \frac{i(1-a)}{M^2}\left[\not{p}_a\not{\partial}\alpha+\not{\partial}\alpha\not{p}_a\right] + O(\alpha^2)\right\},$$

$$\exp\left\{-(\tilde{\not{p}}_a[A] + i(1-a)\not{\partial}\alpha)^2/M^2\right\} = \exp\left\{-\frac{\tilde{\not{p}}_a^2}{M^2} - \frac{i(1-a)}{M^2}\left[\tilde{\not{p}}_a\not{\partial}\alpha+\not{\partial}\alpha\tilde{\not{p}}_a\right] + O(\alpha^2)\right\} \tag{4.48}$$

To this end, we note

$$e^{A+B} = e^{A}\left\{1+B + \frac{1}{2}[B,A] + O(B^2) + \text{terms containing multiple commutators}\right\}, \tag{4.49a}$$

$$e^{A'+B'} = \left\{(1+B' - \frac{1}{2}[B',A']) + O(B'^2) + \begin{matrix}\text{terms containing} \\ \text{multiple commutators}\end{matrix}\right\} e^{A}. \tag{4.49b}$$

Letting $A = -\frac{\tilde{\not{p}}_a^2}{M^2}$; $B = -\frac{i(1-a)}{M^2}[\tilde{\not{p}}_a\not{\partial}\alpha + \not{\partial}\alpha\, \tilde{\not{p}}_a]$ in Eq. (4.49a) and $A' = -\frac{\not{p}_a^2}{M^2}$, $B' = -\frac{i(1-a)}{M^2}[\not{p}_a\not{\partial}\alpha + \not{\partial}\alpha\, \not{p}_a]$ in Eq. (4.49b) and using these results in Eqs. (4.41), one obtains

$$W'^{M}_{\nu}[A] = \frac{1}{W[A]} \int D\psi\, D\bar{\psi}\, e^{S}\, \bar{\psi}(x)\, \exp\left[-\frac{\not{D}_a^2}{M^2}\right] \left\{1+B+\frac{1}{2}[B,A]\right\}$$

$$\times\, \gamma_\nu \gamma_5 \left\{1+B' - \frac{1}{2}[B',A']\right\} \exp\left[\frac{\not{D}_a^2}{M^2}\right] \psi(x)$$

$$+ \text{ multiple commutator terms } + O(\alpha^2). \qquad (4.50)$$

As shown later (see Appendix F) the multiple commutator terms do not contribute in the limit $M \to \infty$. Thus, from Eqs. (4.44) and (4.47), we have,

$$+\frac{1}{e} \int d^4y\, \alpha(y)\, \partial^\mu \frac{\delta W^M_\nu}{\delta A_\mu(y)}$$

$$= \frac{1}{W[A]} \int D\psi\, D\bar{\psi}\, e^{S}\, \bar{\psi}(x)\, \exp\left[-\frac{\not{D}_a^2}{M^2}\right] \left\{B+\frac{1}{2}[B,A]\gamma_\nu\gamma_5\right.$$

$$\left. +\, \gamma_\nu\gamma_5\, (B'-\frac{1}{2}[B',A'])\right\} \exp\left[-\frac{\not{D}_a^2}{M^2}\right] \psi(x). \qquad (4.51)$$

Carrying out the ψ, $\bar{\psi}$ integration as was done for the axial-vector case, or equivalently directly using,

$$-\frac{1}{W[A]} \int D\psi\, D\bar{\psi}\, e^{S}\, \psi(x)\, \bar{\psi}(z) = \tilde{G}(x,z) = \langle x \mid \frac{1}{i\not{D} - m_o} \mid z \rangle, \qquad (4.52)$$

we obtain

$$\frac{1}{e} \int d^4y\, \alpha(y)\, \partial^\mu \frac{\delta W^M_\nu}{\delta A_\mu(y)}$$

$$= \mathrm{Tr}\int d^4z\, \delta^4(z-x)\, \exp\left[-\frac{\not{D}_{ax}^2}{M^2}\right] \left\{(B+\frac{1}{2}[B,A]\gamma_\nu\gamma_5 +\right.$$

$$\left.\gamma_\nu\gamma_5\, (B'-\frac{1}{2}[B',A'])\right\} \exp\left[-\frac{\not{D}_{ax}^2}{M^2}\right] G(x,z) \qquad (4.53)$$

We evaluate this as was done for the chiral case by using

$$G(x,z) = G_o(x,z) + e \int G_o(x,y) \not{A}(y) G_o(y,z) d^4y + \ldots. \tag{4.54}$$

Details of the calculation are given in Appendix F. Here, we simply state the results :

$$\partial_y^\mu \frac{\delta W_\nu(x)}{\delta A_\mu(y)} = \frac{(1-a^2)ie^2}{16\pi^2} \varepsilon_{\nu\mu\rho\sigma} F^{\mu\rho} \partial_x^\sigma \delta^4(x-y). \tag{4.55}$$

Comparing this with Eq. (4.6b), we obtain the same value for $\beta = - a^2$ as in Eq. (4.34).

We thus obtain the family of anomalies of Eqs. (4.6) in the path integral formulation.

Finally, we comment on one possibly undesirable feature of our results and a way to rectify it. Eq.(4.34) says that if 'a' is real, β is always negative, in contrast to the result in ordinary perturbation theory where β can take any real value. And 'a' is required to be real for $\not{D}_a$ to be hermitian for real gauge fields. We could generalize the result of Eq. (4.6) for imaginary values of 'a' as follows. W_ν [A] is a function of fields which could be continued to purely imaginary A_μ. Then $\not{D}_a$ will be hermitian for purely imaginary 'a'. Hence we could assume reality of eigenvalues and completeness of its eigenfunctions. The entire procedure will then go through leading to the result of Eqs. (4.6a) and (4.6b) for purely imaginary A_μ . We then <u>define</u> W_ν [A] for real A_μ and imaginary 'a' by the <u>same formal expression</u> and thus Eqs. (4.6a) and (4.6b) become valid with $\beta = -a^2$ holding for arbitrary real β.

CHAPTER - V

A COMPACT ARGUMENT FOR FAMILY OF ANOMALIES

5.1 INTRODUCTION

In the previous three chapters, the family structure of anomalies in $(QED)_{2,4}$ was established within the path-integral framework by doing explicit calculations. In four dimensions, the family of anomalies that was established, was in terms of a regularized axial-vector current, viz.,

$$\partial^{\nu} W_{\nu}[A] = 2im_o\, W_p[A] + \frac{ie^2(1-\beta)}{16\pi^2}\, F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{5.1a}$$

and

$$\partial^{\mu}_{y} \frac{\delta\, W_{\nu}\, [A,x]}{\delta\, A_{\mu}\, (y)} = \frac{ie^2\, (1+\beta)}{16\pi^2}\, \epsilon_{\nu}{}^{\mu\rho\sigma}\, F_{\mu\rho}(x)\, \partial^{x}_{\sigma}\, \delta^4(x-y). \tag{5.1b}$$

(The reason for establishing the family structure in this form was given in section (4.1) of chapter IV). By explicit (and laborious !) calculations for the terms contributing to right-hand side of equations (5.1) the results were obtained and it was seen that $\beta = -a^2$, a being the free parameter in operator $\not{D}_a$ which was used to define and regularize $\langle J_{\nu}{}^{A} \rangle$.

In this chapter, we present a much *simpler* argument that proves the *family structure* of Eqs. (5.1) [20]. We show that as far as the family structure of anomalies is concerned, it could be arrived at by an examination of W_{ν} and its properties, that is far

more brief than the calculations of chapter IV [19]. Such an argument will also come in handy when we give a very general formulation of anomalies in QED [22] in chapter VI. In the non-abelian case too, when we deal with the question of 'covariant' and 'consistent' anomalies in the path-integral framework [23] such an argument becomes an invaluable tool (see chapter VII).

In section (5.2) we give our notations and the definition of regularized axial-vector current. In section (5.3) we prove that if $W^1_\nu \equiv {}_{M\to\infty} (W^M_\nu - W^M_\nu\big|_{a=1})$ is local, the family of anomalies follows. In section (5.4) we show that W^1_ν is indeed local. The parameter β is, of course, not determined in terms of 'a' in this treatment. The proof, given here for the four-dimensional case, goes through in two-dimensions, with a few changes in detail. This is commented upon at the end of section (5.4). In section (5.5) we give the conclusions.

5.2 PRELIMINARY

We consider the Euclidean space Lagrangian for four-dimensional QED:

$$\mathcal{L} = \bar{\psi}\,(i\,\not{D} - m_o)\,\psi\,, \tag{5.2}$$

where $\not{D} = \gamma^\mu D_\mu = \gamma^\mu\,(\partial_\mu + ieA_\mu)$ and A_μ is a real abelian external field. We shall mainly use the notations of Ref. 7. $\gamma^k (k = 1,2,3)$ and $\gamma^4 = i\gamma^0$ are all antihermitian. $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ is hermitian. $\epsilon_{1234} = \epsilon^{1234} = 1$. The metric is $g_{\mu\nu}$ = Diag. (-1,-1,-1,-1).

We consider the path-integral in Euclidean space

$$W[A] = \int D\psi\, D\bar{\psi}\; e^{S}, \tag{5.3}$$

which will be defined in terms of the eigenfunctions of the operator $\not{D}_a = \not{\partial} + iea\not{A}$.

As in chapter IV, we define the regularized axial-vector current as[1]

$$J_\nu^{AM}(x) = \bar{\psi}(x) \exp\left[-\frac{\not{D}_a^2}{M^2}\right] \gamma_\nu \gamma_5 \exp\left[-\frac{\not{D}_a^2}{M^2}\right] \psi(x), \tag{5.4}$$

and connected, regularized Green's functions with external gauge boson lines of J_ν^A are generated by

$$W_\nu^M[A] = \frac{1}{W[A]} \int D\psi\, D\bar{\psi}\; J_\nu^{AM}(x) \quad e^{S}. \tag{5.5}$$

Under "parity" transformations $\psi \longrightarrow \gamma^4 \psi$, $\bar{\psi} \rightarrow -\bar{\psi}\gamma^4$, $A_i \rightarrow -A_i$ $(i = 1,2,3)$ and $A_4 \longrightarrow A_4$, J_ν^{AM} transforms as a pseudo-vector and so does $W_\nu^M[A]$.

---

[1] Our derivation can be looked upon directly as the evaluation of $\langle J_\nu^{AM}(x)\rangle$ defined below via Eq. (5.16) without reference to path integral formulation. However, the definition given below finds a *natural* interpretation within Fujikawa's path-integral formulation wherein ψ and $\bar{\psi}$ are expanded in terms of eigenfunctions of $\not{D}_a$. In other words, in obvious notations [7],

$$J_\nu^A(x) = \sum_n \sum_m \bar{b}_n\, a_m\, \varphi_n^\dagger(x)\, \gamma_\nu \gamma_5\, \varphi_m(x)$$

$$\longrightarrow J_\nu^{AM}(x) = \sum_n \sum_m \bar{b}_n\, a_m\, e^{-\lambda_n^2/M^2}\, e^{-\lambda_m^2/M^2}\, \varphi_n^\dagger(x)\gamma_\nu\gamma_5\varphi_m(x)$$

Thus our regularization amounts to introducing a Fujikawa regularization function $\exp(-\lambda^2/M^2)$, for each summation.

5.3 FORM OF W_ν AND THE FAMILY STRUCTURE

We shall prove the result for the family of anomalies in two stages, in this and the next section. In this section, we shall consider the expectation value of the axial-vector current, $W_\nu[A]$, and consider contributions to it arising specifically from the gauge-variant regularization (and thus involving (a-1) or its powers). We shall show then that if this piece, called $W_\nu^1[A]$, is a local functional of A_μ, the result for the family of anomalies follows. Then, in the next section, we shall prove that $W_\nu^1[A]$ is indeed local, completing the proof.

We consider,

$$W_\nu [A] = \lim_{M\to\infty} \frac{1}{W[A]} \int D\psi\, D\bar{\psi}\, e^S\, \bar{\psi}(x) \exp\left[-\frac{\bar{\not{D}}_a^2}{M^2}\right] \gamma_\nu\gamma_5 \times \exp\left[-\frac{\not{D}_a^2}{M^2}\right] \psi(x) \qquad (5.6)$$

$$\equiv \lim_{M\to\infty} \langle J_\nu^{AM}(x) \rangle .$$

We split $W_\nu[A]$ into two parts :

$$W_\nu[A] = W_\nu{}^{o}[A] + W_\nu{}^{1}[A]. \qquad (5.7)$$

Here

$$W_\nu{}^{o}[A] = W_\nu[A] \Big|_{a=1}, \qquad (5.8)$$

is the expectation value of the axial-vector current in the gauge-invariant regularization (a = 1). The gauge invariant

regularization is, of course, the one already studied by Fujikawa [7] in a somewhat different form and gives the well known result for the chiral anomaly :

$$\partial^{\nu} W_{\nu}{}^{o}[A] = 2\ im_{o}\ W_{P}[A] + \frac{ie^2}{8\Pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \tag{5.9}$$

with

$$W_P[A] \equiv \lim_{M\to\infty} \langle \bar{\psi}(x) \exp\left[-\frac{\overleftarrow{\not{D}}_a^2}{M^2}\right] \gamma_5 \exp\left[-\frac{\not{D}_a^2}{M^2}\right] \psi(x) \rangle \tag{5.10}$$

[The result of Eq. (5.9) should contain $W_P^o[A] = W_P[A]\big|_{a=1}$. However as shown at the end of Sec.(5.4), $W_P[A]$ is independent of a.]

The result for the gauge invariance of $W_{\nu}{}^{o}[A]$ is expressed as

$$\partial^{\mu} \frac{\delta}{\delta A_{\mu}} W_{\nu}{}^{o}[A] = 0. \tag{5.11}$$

Equations (5.9) and (5.11) are just the special case of the anomaly equations (5.1) at $\beta = -1$. Our aim, in this work, is to generalize the above results for an arbitrary value of 'a', which then establishes the equations characterizing the family of anomalies.

Now, we shall establish a theorem relating the <u>*form*</u> of $W_{\nu}{}^{1}[A]$ to the family <u>*structure.*</u>

Theorem I : If $W_\nu{}^1[A]$ is of the form

$$W_\nu{}^1[A] = f(a)\, \epsilon_\nu{}^{\mu\lambda\sigma} A_\mu\, \partial_\lambda\, A_\sigma\,, \tag{5.12}$$

then the result of the family of anomalies of Eqs. (5.1) follows immediately with

$$\beta = -1 + \frac{8\,\Pi^2 i\, f(a)}{e^2}\,. \tag{5.12a}$$

Proof : It is easily seen from Eqs. (5.7), (5.9) and (5.12) that

$$\partial^\nu W_\nu[A] = 2\, i m_o\, W_P[A] + \left[\frac{ie^2}{8\Pi^2} + \frac{f(a)}{2} \right] F_{\mu\nu}\, \tilde{F}^{\mu\nu}, \tag{5.13a}$$

and from Eqs. (5.7), (5.11) and (5.12) that

$$\partial^\mu \frac{\delta}{\delta A_\mu(y)} W_\nu\,[A] = -\frac{1}{2} f(a)\, \epsilon_\nu^{\mu\lambda\sigma} F_{\mu\lambda}(x)\, \partial_\sigma^x\, \delta^4(x-y)\,. \tag{5.13b}$$

Eqs. (5.13a) and (5.13b) are identical to Eqs. (5.1) with β given by Eq. (5.12a). Hence etc.

We wish to make a few remarks. From Eqs (5.7), (5.8) and (5.12) it follows that

$$f(a)\Big|_{a=1} = 0\,, \tag{5.14}$$

and thus at $a = 1$, $\beta = -1$ as seen from Eq. (5.12a), the value corresponding to no vector anomaly.

The result of theorem I states an important point, viz., it is only the form of $W_\nu^1\,[A]$ which is needed to prove the family structure and not the detailed expression. In particular, f(a)

which is determined by a laborious calculation in chapter IV, is not needed at all for this purpose. (For completeness we shall state the result for f(a) :

$$f(a) \quad = \quad - \frac{ie^2}{8\Pi^2} \quad (1-a^2) \tag{5.15}$$

yielding $\beta = -a^2$).

Next we shall show that the form posited in Eq. (5.12) is indeed a result of the *locality* of $W_\nu{}^1[A]$ alone.

Theorem II : If $W_\nu{}^1[A]$ is a local functional of A_μ , then it has the form of Eq. (5.12):

$$W_\nu{}^1[A] = f(a) \, \epsilon_\nu{}^{\mu\lambda\sigma} A_\mu \, \partial_\lambda \, A_\sigma \; .$$

Proof : $W_\nu{}^1[A]$ is a pseudovector of canonical dimension three. As is easily verified, $\epsilon_\nu{}^{\mu\lambda\sigma} A_\mu \partial_\lambda A_\sigma$ is the only functional of A_μ that is local and a pseudovector and has dimension three or less. Q.E.D.

In the next section, we shall prove the locality of $W_\nu{}^1[A]$ given by Eq. (5.7). It will turn out to be necessary to examine W_ν somewhat closely. But unlike chapter IV, no detailed calculation will be needed.

5.4 LOCALITY OF W_ν^1

We shall first cast W_ν [A] in a form from which the structure of terms contributing to W_ν^1 [A] can be read. We shall then put down the conditions that determine which terms contribute to W_ν^1 [A]. We shall discuss when a term contributing to W_ν^1 [A]

is local. We shall find it necessary to consider the infrared behavior of the integrals involved. These will ultimately lead to the result that W^1_ν [A] is local, which as shown in the previous section, is sufficient to prove the family structure.

We rewrite W_ν [A] as

$$W_\nu\ [A] = -\lim_{M\to\infty} \mathrm{Tr} \int d^4z\ \delta^4(x-z)\ \exp\left[-\frac{\not{\mathfrak{D}}^2_{ax}}{M^2}\right] \gamma_\nu\gamma_5 \exp\left[-\frac{\not{D}^2_{ax}}{M^2}\right]$$

$$\times \frac{1}{W[A]} \int D\psi\ D\bar{\psi}\quad e^S\ \psi(x)\ \bar{\psi}(z)$$

$$\equiv \lim_{M\to\infty} \mathrm{Tr} \int d^4z\ \delta^4(x-z)\ \exp\left[-\frac{\not{\mathfrak{D}}^2_{ax}}{M^2}\right] \gamma_\nu\gamma_5 \exp\left[-\frac{\not{D}^2_{ax}}{M^2}\right] G(x,z;A). \tag{5.16}$$

Here G(x, z; A) is the fermion propagator in presence of external field A_μ, and has the expansion

$$G(x,z;A) = G_o(x,z) + e \int G_o(x,y)\ \not{A}(y)\ G_o(y,z)d^4y + \ldots\ldots \tag{5.17}$$

We express

$$\delta^4(x-z)\ \exp\left[-\frac{\not{\mathfrak{D}}^2_{ax}}{M^2}\right] = \int \frac{d^4k}{(2\pi)^4} e^{ik.(x-z)}\quad 1.\exp\left[-\frac{(\not{\mathfrak{D}}_{ax} - i\not{k})^2}{M^2}\right] \tag{5.18}$$

and

$$\exp\left[-\frac{\not{D}^2_{ax}}{M^2}\right] G_o(x,y) = \int \frac{d^4k_1 e^{-ik_1.(x-y)}}{(2\pi)^4} \exp\left[-\frac{(\not{D}_{ax}-i\not{k}_1)^2}{M^2}\right].1\frac{+1}{\not{k}_1-m_o}\ . \tag{5.19}$$

The nth-order term in Eq. (5.17) is,

$$e^{n-1} \int \frac{d^4k_1\ d^4k_2 \ldots\ldots d^4k_n}{(2\pi)^{4n}} \frac{1}{\not{k}_1 - m_o} \tilde{\not{A}}\,(k_1-k_2)\, \frac{1}{\not{k}_2 - m_o} \tilde{\not{A}}\,(k_2-k_3) \ldots$$
$$\ldots \tilde{\not{A}}\,(k_{n-1} - k_n)\, \frac{1}{\not{k}_n - m_o}\, e^{-ik_1 . x\, +\, ik_n . z} \quad . \tag{5.20}$$

This n^{th} term in G contributes to the right-hand side of Eq.(5.16); this contribution, upto factors, is expressed, using Eqs. (5.18) and (5.19) as

$$\mathrm{Tr} \int d^4z \int d^4k\ d^4k_1 \ldots d^4k_n\ \ e^{ik.(x-z)\ -\ ik_1 . x\ +\ ik_n . z}\ \ e^{\left[\frac{k^2}{M^2} + \frac{k_1^2}{M^2}\right]}$$
$$\times\ 1.\left\{\exp - \frac{1}{M^2}\left[\overleftarrow{\not{D}}^2_{ax} - 2ik.\overleftarrow{D}_a\right]\right\} \gamma_\nu\gamma_5\ \exp\left\{- \frac{1}{M^2}\left[\not{D}^2_{ax} - 2ik_1 . D_{ax}\right]\right\} . 1$$
$$\times\ \frac{1}{\not{k}_1 - m_o}\ \tilde{\not{A}}\,(k_1-k_2)\ \frac{1}{\not{k}_2 - m_o}\ \tilde{\not{A}}\,(k_2-k_3)\ \ldots\ldots\ \tilde{\not{A}}\,(k_{n-1}-k_n)\ \frac{1}{\not{k}_n - m_o}\ . \tag{5.21}$$

We shall now introduce momentum variables

$$q_1 = k-k_1,\quad q_2 = k_1-k_2\ ,\ \ldots\ldots, q_n = k_{n-1} - k_n\ , \tag{5.22}$$

so that upto an overall factor, the expression (5.21) becomes

$$\mathrm{Tr} \int d^4z \int d^4q_1 \ldots d^4q_n\ \ e^{iq_1 . (x-z)\ -\ i(q_2 + \ldots + q_n) . z}$$
$$\times \int d^4k\ \exp\left[\frac{k^2}{M^2} + \frac{(k-q_1)^2}{M^2}\right]\ 1.\left\{\exp - \frac{1}{M^2}\left[\overleftarrow{\not{D}}^2_{ax} - 2ik.\overleftarrow{D}_{ax}\right]\ \gamma_\nu\gamma_5\right.$$
$$\left.\exp\left\{- \frac{1}{M^2}\left[\not{D}^2_{ax} - 2i\ (k-q_1) . D_{ax}\right]\right\}\ . 1\ \ \frac{1}{(\not{k}-\not{q}_1) - m_o}\right.$$
$$\times\ \tilde{\not{A}}(q_2)\ \frac{1}{(\not{k}-\not{q}_1-\not{q}_2) - m_o}\ \tilde{\not{A}}(q_3) \ldots\ldots \tilde{\not{A}}(q_n)\ \frac{1}{(\not{k}-\not{q}_1 - \ldots - \not{q}_n) - m_o} . \tag{5.23}$$

We now imagine scaling $k \longrightarrow Mk$ everywhere in expression (5.23). The relevant k integral then becomes, upto factors,

$$M^{4-n} \int d^4k\, e^{2k^2} \exp\left[-\frac{2k.q_1}{M} + \frac{q_1{}^2}{M^2}\right] \times 1.\, \exp\left[-\frac{\tilde{\not{D}}^2_{ax}}{M^2} + \frac{2ik.\tilde{D}_{ax}}{M}\right]\Bigg\} \gamma_\nu \gamma_5$$

$$\exp\left\{-\frac{\not{D}^2_{ax}}{M^2} + \frac{2i(k-q_1/M).D_{ax}}{M}\right\}.1\, \frac{1}{\not{k}-(\not{q}_1+m_o)/M}\, \tilde{\not{X}}(q_2)\,.......\,\tilde{\not{X}}(q_n)$$

$$\times \frac{1}{\not{k}-(\not{q}_1+...+\not{q}_n+m_o)/M}\,. \tag{5.24}$$

The general form of a typical term is

$$M^{4-n}\, d^4k\, \mathrm{Tr}\left[\gamma_\nu\gamma_5\left(\frac{\not{D}^2_a}{M^2}\right)^\alpha\left(\frac{k'.D_a}{M}\right)^\beta\left(\frac{\not{k}''+m_o/M}{k''^2-m_o^2/M^2}\right)^n (\not{X})^{n-1}\right]\exp\left[-\frac{2k.q_1}{M}\right], \tag{5.25}$$

$\alpha,\ \beta,\ n-1 = 0,1,2,......,$

where we have paid no attention to the ordering of γ-matrices and k' refers to either k and/or $k - \frac{q_1}{M}$ and k'' refers to

$$k - \frac{q_1}{M}\,,\quad k - \frac{q_1+q_2}{M}\,,\ ....,\ k - \frac{q_1+....+q_n}{M}\,.$$

In order that (5.25) be nonzero, the trace of γ matrices must not vanish. This requires that there are at least four γ matrices with γ_5, so that

$$2\alpha + 2n \geq 4\,.$$

The *leading* M power is seen to be

$$\Delta_{max} = 4-n - 2\alpha - \beta \ . \tag{5.26}$$

We now imagine expanding (5.25) in powers of q_i and m_o, upto a sufficiently many but *finite* number of terms.

$\left[\text{Example} = \dfrac{1}{x^2-m_o^2} = \dfrac{1}{x^2} + \dfrac{m_o^2}{x^4} + \dfrac{m_o^4}{x^4(x^2-m_o^2)}\right]$. The actual M power of a typical term obtained from such an expansion will be less than Δ_{max} by a non-negative integer η (≥ 0) so that

$$\Delta = 4-n - 2\alpha - \beta - \eta \ . \tag{5.27}$$

The overall power of k of *this* specific term, which determines the convergence in the infrared region, is

$$\Delta' = 4 + \beta - n - \eta' \tag{5.28}$$

where η' is an integer. An inspection of (5.25) will convince the reader that

$$\eta - \eta' \geq 0. \tag{5.29}$$

Now, each term in the expansion of (5.25) carries a non-negative power of 'a' arising solely from ϕ_a. A term will contribute to $W_\nu^1 = W_\nu - W_\nu \big|_{a=1}$, only if it carries a positive power of a. This requires that either $\alpha > 0$ or $\beta > 0$ so that

$$\alpha + \beta \geq 1 \tag{5.30}$$

We now focus our attention to terms that contribute to W_ν^1 as $M \to \infty$. For such a term $\Delta \geq 0$. We then find that for such a term

$$\begin{aligned}\Delta' &= (4-2\alpha-\beta-n-\eta) + 2(\alpha+\beta) + (\eta-\eta') \\ &= \Delta + 2(\alpha+\beta) + (\eta-\eta') \\ &> 0, \end{aligned} \tag{5.31}$$

where a use of Eqs. (5.29) and (5.30) has been made. We shall now prove a theorem which implies that the condition of Eq. (5.31) is indeed sufficient for locality of W^1_ν [A].

Theorem : If $\Delta' >$ 0 for each term contributing to $W^1_\nu[A]$ from the right-hand side of Eq. (5.23), then W^1_ν [A] is local.

Proof : First we note that there are only a finite number of terms in the expansion of (5.25) and of (5.17) that contribute as $M \to \infty$ [See Eq. (5.27)]. As $\Delta' > 0$ for each of the terms contributing to W^1_ν , the k integrals are infrared convergent and hence well defined. The result is a polynominal in $q_1 \ldots . q_n$ and $A(q_2)$ ..... $A(q_n)$ and $A(x)$ and its derivatives. Each contribution to $W^1_\nu[A]$ from expression (5.23) can be broken up into terms of the form

$$\int d^4z \int e^{iq_1\cdot(x-z)} P_1(q_1)\, d^4q_1 \int d^4q_2\, e^{-iq_2\cdot z} A(q_2)\, P_2(q_2) \times \ldots.$$
$$\times \int d^4q_n\, e^{-iq_n\cdot z} A(q_n)\, P_n(q_n) \times \text{local polynomial in } A(x), \tag{5.32}$$

where P_1, ... P_n are polynomials in q_1, ... q_n, respectively. Hence expression (5.32) is seen to be local. Now W^1_ν [A] consists of contributions with $\Delta' > 0$ from terms of the form of (5.23) and hence is itself local.

This together with theorems I and II of the previous section completes the proof of the family structure of anomalies.

The argument of Sec (5.4) we gave, applies equally well to $W_p[A]$ instead of $W_\nu[A]$. Hence the a-dependent contributions to $W_p[A]$ must be local and also pseudoscalars of dimension three. But there are no such contributions. Hence $W_p[A] = W_p^o[A]$.

A final comment : Locality implies that W_ν^1 has the structure of Eq. (5.12) and powercounting and dimensions convince one that W_ν^1 [A] is of $O(M^o)$ and hence finite as $M \to \infty$.

We note that our proofs would hold equally well in two dimensions with the following changes: W_ν^1 [A] is of the form $\tilde{f}(a)\epsilon_\nu{}^\mu A_\mu$, and one has to replace 4 by 2 in Eqs. (5.25), (5.27), (5.28) in particular.

Our proof of family structure can be generalized greatly and could be applied [22], say, to point-splitting regularization (after certain modifications to take into account the lack of manifest Lorentz covariance). (see section (6.7) of chapter VI).

5.5 CONCLUSION

In this chapter, we have shown the existence of family structure of anomalies in $(QED)_4$ by establishing locality of W_ν^1. This was done by examining the parameter dependent contributions to W_ν. No detailed calculations were needed. The proof for the

family structure was seen to hold, after appropriate modifications, in two dimensions also. Arguments, along the lines of those used in this chapter for establishing locality of parameter dependent contributions to the expectation values of currents, will be used in the following two chapters also, to establish the results there.

CHAPTER - VI

A GENERAL FORMULATION OF ANOMALIES IN QED IN PATH-INTEGRAL FORMULATION

6.1 INTRODUCTION

In this chapter we give a very general formulation of anomalies and their family structure in QED in path-integral formulation which is based on the definition of the axial-vector current regularized in a very general manner [22]. So far in the thesis, our derivation of family structure of anomalies in the path-integral framework has made use of a somewhat specific definition of regularized currents (see equations (3.4), (3.5), (4.4)) and their expectation values. Such definitions have been based on the use of eigenfunctions and eigenvalues of the operator $D\!\!\!/_a \equiv \partial\!\!\!/+iea\!\!\!/$. These definitions proved sufficient in leading to sensible and *well-defined* derivations of the chiral and vector anomalies in $(QED)_{2,4}$. With a view to generalize the definition, we note the two main ingredients which go into defining expectation values of regularized currents:

(1) An operator with a complete set of eigenfunctions for expanding the fermionic fields and for defining the path-integral.

(2) A regularizing function for defining ill-defined, unregularized quantities. Such a function is defined in terms of the eigenvalues of the operator which is used to define the path-integral.

In this chapter we show that the formalism developed in the previous chapters for the special case (of an operator $\not{D}_a$ and a special regularizing function $e^{-\not{D}_a^2/M^2}$), is powerful enough to allow a very wide generalization, with only some technical modifications needed for the details. The motivations for generalization are clarified by the following observations, and results to be established in this chapter.

(1) As was pointed out in chapter I, insofar as the definition of the path-integral measure is concerned, we could expand the fermion fields ψ and $\bar{\psi}$ in <u>any</u> bases. For example, one could expand ψ in terms of any complete set of basis functions

$$\psi(x) \quad = \sum_n a_n \phi_n(x) \tag{6.1a}$$

Here $\phi_n(x)$ could be the complete set of eigenfunctions of a local hermitian opertor X[1]. One can look upon the fermionic functional integral as one in which ψ and $\bar{\psi}$ are independent Grassmann variables. One can then expand $\bar{\psi}$ in terms of left eigenfunctions of possibly another operator Y

$$\bar{\psi}(x) \quad = \sum_m \bar{b}_m \chi_m^\dagger (x), \tag{6.1b}$$

and then define $D\psi D\bar{\psi} = \prod_m d\bar{b}_m \prod_n da_n$. Further, in defining

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[1] X can also be chosen to be a nonhermitian operator having a complete set of eigenfunctions. Our discussion applies also to a restricted class of nonhermitian operators. For further details see Sec (6.4B).

unregulated ill-defined quantities, one could introduce possibly independent regularizing functions $f(\lambda_n)$ and $g(\lambda'_m)$ in terms of the eigenvalues λ_n and λ'_m of X and Y, respectively. (See for example, Eq. (6.3) below). Further, the operators X and Y and regularizing functions could be chosen widely. All these definitions of the path-integral and associated ill-defined quantities are, from a purely formal point of view, on an equal footing (in absence of physical constraints). Therefore, an important result like derivation of anomalies and family structure should not, in our opinion, depend crucially on specific choices of operators. As we shall see in this chapter, anomaly derivations in path-integral formulation can be carried out by using very general basis to define the path-integral. The family of anomalies derived in the previous chapters is not crucially dependent on the choice of operator as $\not{D}_a$. The *one* parameter family structure in QED holds for a very general X (which could depend on many parameters), of which $\not{D}_a$ happens to be the simplest special case, sufficient to get family structure. This is discussed in sections (6.3) and (6.4).

(2) Another important aspect of the work in this chapter is a detailed mathematical formulation to see how ambiguities in anomaly formulations (which manifest as arbitrariness in anomaly expressions) arise, in the path-integral formulation, from the arbitrariness in the choice of basis for defining the path-integral measure. In the context of QED, anomaly expressions are arbitrary due to the presence of parameter β in family equations. β is related to the choice of operator X to define the path-integral. (See, for example equations (6.6) and (6.20)

below).

(3) In the literature, QED anomalies have been derived using various methods like Feynman diagrammatic methods [See first of Ref. 2], point splitting methods for regularizing currents [34], path-integral formulation of Fujikawa [7] etc. We shall see that all such derivations *can* be looked upon as special cases of our works in that these derivations are *equivalent* to ours for *special choices* of regularizing operators X, Y and regularizing functions. This is made clear in sections (6.6) and (6.7).

Finally, we would like to emphasize another aspect of this work. Until Fujikawa's derivation of chiral anomaly, regularizations used in QFT have been generally based on momentum space, (apart from the coordinate space regularizations such as point-splitting, lattice regularizations) and detailed rigorous results have been derived for such momentum space regularizations. Our study, in this chapter, can be looked upon as a study of a wide class of new field-dependent regularizations (using general operator X) done with the example of chiral anomaly; the momentum space regularizations being a special case with $X = \not{\partial}^2/M^2$. The procedures used in establishing theorems in this chapter may have a general utility in the context of such field-dependent regulators. [It should be pointed out that field-dependent regulators <u>are</u> nontrivially different from momentum space regularizations, e.g., quantum equations of motion are nontrivially altered [52,53]].

In section (6.2) we give our notations and also give explicitly the definition of the regularized axial-vector current.

In section (6.3) we establish the family structure of anomalies based on this very general definition of the regularized axial-vector current. We state the assumptions on the form of the operators used for defining the path integral. In section (6.4), we discuss in detail the restrictions on the regularizing functions and operators, which are necessary for our proof of family structure to go through. In section (6.5) we give a *general* and *well-defined* derivation of chiral anomaly equation where a very general operator X is used to define the path-integral. Use is also made of a generalized W-T identity. Anomaly terms arise in a well-defined and regularized form, naturally, and further, they are shown to arise both from jacobian and action sources. Sections (6.6) and (6.7) are devoted to comparisons with other known derivations of the QED anomalies. In section (6.8) we make some concluding remarks.

6.2 PRELIMINARY

A. Notations

We shall mainly work in the context of $(QED)_4$ though our results can be generalized to $(QED)_2$ easily.

We consider the four-dimensional Euclidean lagrange density for a Dirac fermion ψ :

$$\mathcal{L} = \bar{\psi}\,(i\,\not{D} - m_o)\psi \tag{6.2}$$

where $\not{D} = \partial^{\mu}(\partial_{\mu} + ieA_{\mu})$. A_{μ} is a real abelian external field. The notations used are those already defined in section (4.2) of chapter IV.

B. *Regularized axial-vector current: choice of operators and regularizing functions*

The regularized axial-vector current is defined as follows:

$$J_{\mu}^{AM}(x) = \bar{\psi}(x)\, g(\overleftarrow{Y})\, \gamma_{\mu}\gamma_{5}\, f(X)\, \psi(x) \qquad (6.3)$$

[Here $\overleftarrow{Y}$ is defined by $(Y^{\dagger}\chi)^{\dagger} \equiv \chi^{\dagger}\,\overleftarrow{Y}$]

The operator X is local, dimensionless and is a finite polynomial in A_{μ} (e.g. $X = \frac{\not{D}^2}{M^2}$). We shall assume that X depends only on one mass parameter M and finally we shall let $M \longrightarrow \infty$. We choose X to have the form[2]

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[2]This form of the operator seems unusual for two reasons. (a) The large dimensionful parameter M is included in X itself. (b) X is allowed to contain *different* powers of M. The first has been done for reasons of technical simplicity in writing but the second needs explanation. Our choice for X certainly includes operators of the form $\frac{G(\partial,A,\ldots.)}{(M^2)^n}$ where G is a dimension 2n operator that is indpendent of M [such as $(\not{D})^{2n}$]. In this case ϕ_ns are eigenfunctions of G and are independent of M. But we can also include cases in which X contains operators of two or more canonical dimensions and then (only), in $X(M)\,\phi_n(M) = \lambda_n(M)\phi_n(M)$, ϕ_n and λ_n are M dependent. While this is unusual, we have included it because in the same argument we can include all possibilities simultaneously.

$$X = \sum_{i=1}^{N} \frac{\tilde{F}_i[\partial, A, \partial A, \ldots\ldots]}{M^{D_i}} \tag{6.4}$$

where $\tilde{F}_i[\partial, A, \partial A, \ldots]$ are 4x4 matrix local polynomials in A_μ and derivatives, and

$$\mathrm{Dim}\ (\tilde{F}_i) = D_i \le D_{max} < \infty\ ; \quad N < \infty$$

We shall assume that at A = 0, X = X_o and X_o is a positive semi-definite operator (e.g. $X_o = \frac{\not{\partial}^2}{M^2}$). Further constraints on the form of X are given in section (6.4).

Y is another operator of a similar kind. f and g are certain regularizing functions and their properties are discussed at length in section (6.4).

We shall, generally, define the generating functional of one loop Green's functions with external photon lines as

$$W_\nu^M\ [A;x] = \frac{1}{W[A]} \int D\psi\ D\bar{\psi}\ J_\nu^{AM}(x)\ e^{S[\psi,\bar{\psi},A]} \tag{6.5}$$

This is the starting point for our derivation of family structure as shown in the next section. In evaluating $\partial^\nu W_\nu^M$, the derivation can, when X = Y be interpreted along the lines first formulated by Fujikawa [7] in which the path-integral measure is defined in terms of eigenfunctions of X (for Fujikawa X = $\frac{\not{D}^2}{M^2}$). This is discussed further in Sec. (6.5).

6.3 FAMILY STRUCTURE OF ANOMALIES

In this section we establish the family structure of anomalies given by the following equations:

$$\partial^{\nu} W_{\nu}[A] = 2im_o W_p[A] + \frac{ie^2}{16\pi^2} (1-\beta) F_{\mu\nu} \tilde{F}^{\mu\nu}, \tag{6.6a}$$

$$\partial^{\mu}_{y} \frac{\delta W_{\nu} [A(x)]}{\delta A_{\mu}(y)} = \frac{ie^2(1+\beta)}{16\pi^2} \epsilon_{\nu\mu\rho\sigma} F^{\mu\rho}(x) \partial^{\sigma}_{x} \delta^4(x-y), \tag{6.6b}$$

where

$$W_{\nu}[A] = \lim_{M \to \infty} W_{\nu}^{M} [A], \tag{6.7}$$

and

$$W_p[A] = \lim_{M \to \infty} \langle \bar{\psi}(x) g(\overleftarrow{Y}) \gamma_5 f(X) \psi(x) \rangle \tag{6.8}$$

where β is a parameter depending on the parameters in X and Y. $W_{\nu}^{M}[A]$ is given in equation (6.5). Detailed calculations for family structure were done in chapter IV in the four-dimensional context for the restricted case $X = Y = \not{D}_a^2/M^2$ and $f = g = e^{-X}$ in Eq. (6.3). The calculations were quite laborious and we then followed an indirect route in chapter V to establish the family structure where detailed calculations were not needed. In this section (following the discussion of chapter V), we show that the one parameter family structure holds for very general operators X

and Y which could depend on many parameters in general. The operator $X = \not{D}_a^2/M^2$ is only the simplest special case for which the family structure holds. (Ofcourse, X and Y are not completely arbitrary but must satisfy certain restrictions on their form, to be discussed later in Sec. (6.4). These restrictions are necessary for our proof of family structure to go through). The only restrictions we assume, to begin with, on the regularizing functions are that $f(o) = g(o) = 1$ and that they vanish "sufficiently rapidly" at ∞ so that contributions to $W_\nu[A]$ are ultra-violet convergent. The required behaviour of these functions at ∞ is made precise in section (6.4). We also assume that these functions are a Taylor series in X,Y (see Eq. (6.31)). Therefore, under certain conditions on operators X, Y and functions f and g which will be explored in the next section in greater detail, we derive the family structure of anomalies. We rely heavily on the content of chapter V since the modifications needed in the discussion of chapter V when applied to the present general case, are of a purely technical kind. We begin by stating the assumptions on the form of X (and also of Y).

Consider $X[\partial,A]$ at A=0. We shall assume that $X[\partial,0] \equiv X_o[\frac{\partial^2}{M^2}]$ is a function of $\frac{\partial^2}{M^2}$ only.

We then express

$$X \equiv X_o\left[\frac{\not{D}^2}{M^2}\right] + X'[\partial, A, \partial A, \ldots\ldots] \tag{6.9a}$$

Evidently $X'[\partial,A]|_{A=0}$ vanishes. In a similar manner, we write

$$\overleftarrow{Y} \equiv Y_o \left[\frac{\overleftarrow{\not{D}}^2}{M^2} \right] + \overleftarrow{Y}' \left[\partial, A, \partial A, \ldots\ldots \right] \qquad (6.9b)$$

Now consider first the case, $X' = Y' = 0$. In such a case, X and Y are (possibly complicated) polynomial functions of $\not{D}$ only, a situation very similar to that considered by Fujikawa [7] originally. In this case, we should expect, for arbitrary X_o, Y_o, and f, g (that guarantee regularizaion) the family of anomaly equations with $\beta = -1$, i.e. no vector anomaly. This will be confirmed shortly. This would mean that the deviation from $\beta = -1$ arises only when either or both of X' and Y' are nonvanishing. We would therefore like to split $W_\nu[A]$ into two parts of which a part $W^o_\nu[A]$ arises entirely from X_o and Y_o terms in Eqs. (6.9), and segregate the remaining part. (This indeed has been the motivation for splitting X and Y in the manner of Eqs. (6.9)).

To this end, we define

$$W_\nu[A] = W^o_\nu[A] + W^1_\nu[A] \qquad (6.10)$$

with

$$W^o_\nu[A] = \lim_{M \to \infty} \langle \bar{\psi}(x)\, g_o(\overleftarrow{\not{D}}^2/M^2)\, \gamma_\nu\gamma_5\, f_o(\not{D}^2/M^2)\, \psi(x) \rangle$$

$$\equiv \lim_{M \to \infty} \langle J^{oAM}_\nu(x) \rangle \qquad (6.11)$$

where

$$f_o\,(\not{D}^2/M^2) \equiv f\,[X_o\,(\not{D}^2/M^2)] \qquad (6.12)$$

$$q_o \, (\tilde{\not{D}}^2/M^2) \equiv q \, [Y_o \, (\tilde{\not{D}}^2/M^2)] \tag{6.13}$$

We further define

$$W_p^o \, [A] = \lim_{M \to \infty} \langle \bar{\psi}(x) \; q_o(\tilde{\not{D}}^2/M^2) \; \gamma_5 \; f_o \; (\not{D}^2/M^2) \; \psi(x) \rangle \tag{6.14}$$

$$\begin{aligned}
\partial^\nu \langle J_\nu^{oAM}(x) \rangle = - \langle \; & \bar{\psi}(x) \; [\tilde{\not{D}} + im_o] \; q_o(\tilde{\not{D}}^2/M^2) \; \gamma_5 \; f_o \; (\not{D}^2/M^2) \; \psi(x) \\
& + \bar{\psi}(x) \; q_o(\tilde{\not{D}}^2/M^2) \; \gamma_5 \; f_o \; (\not{D}^2/M^2) \; (\not{D} + im_o) \; \psi(x) \\
& - 2 \; im_o \; \bar{\psi}(x) \; q_o(\not{D}^2/M^2) \; \gamma_5 \; f_o \; (\not{D}^2/M^2) \; \psi(x) \rangle
\end{aligned} \tag{6.15}$$

Now, we observe a result such as

$$\langle \bar{\psi}(x) \; a(\tilde{\not{D}}) \; b(\tilde{\not{D}}) \; \psi(x) \rangle = \langle \bar{\psi}(x) \; a(\not{D}) \; b(\not{D}) \; \psi(x) \rangle \tag{6.16}$$

which is easily seen by expanding $\bar{\psi}$ and ψ in terms of eigenfunctions of $\not{D}$ (which is always possible) and simplifying both sides. Using such a result, Eq. (6.15) simplifies, using Eq. (6.14) to

$$\partial^{\nu} W_{\nu}^{o}[A] = \lim_{M \to \infty} -\langle \bar{\psi}(x) [\not{D} + im_o] \gamma_5 F(\not{D}^2/M^2) \psi(x)$$

$$+ \bar{\psi}(x) \gamma_5 F(\not{D}^2/M^2) (\not{D} + im_o) \psi(x) \rangle$$

$$+ 2 im_o W_p^o [A], \tag{6.17}$$

where $F(x) = g_o(x) f_o(x)$. The first two terms on the right-hand side of Eq. (6.17) are precisely those evaluated by Fujikawa [7] except that the exponential regulator e^{-x} is replaced by another function F(x). But as Fujikawa [7] has shown, the result, as $M \to \infty$, does not depend on F provided certain conditions on F are satisfied. This is shown in sec. (6.4) to be the case for f and g that are an acceptable pair of regularizing functions. (see also Eqs. (6.109) and (6.110) below). Hence,

$$\partial^{\nu} W_{\nu}^{o}[A] = 2 im_o W_p^o [A] + \frac{ie^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}. \tag{6.18a}$$

From gauge invariance of $W_{\nu}^{o}[A]$ it also follows that

$$\partial_y^{\mu} \frac{\delta}{\delta A_{\mu}(y)} \overset{o}{W}_{\nu}[A(x)] = 0. \tag{6.18b}$$

These equations coincide with Eqs. (6.6) for $\beta = -1$ except that $W_p[A]$ in Eq. (6.6a) is replaced by $W_p^o[A]$ in Eq.(6.18a). As seen at the end of this section, $W_p[A] = W_p^o[A]$. So Eqs. (6.18) give the usual anomaly equations at $\beta = -1$. It is thus seen that the family structure of anomalies for $W_{\nu}[A]$ would arise from the part $W_{\nu}^{1}[A]$ of $W_{\nu}[A]$, defined by Eq. (6.10). Hence, as in chapter V, it is this part that we study in some detail. As in chapter V, we shall see that the family structure by itself (and

not the value of β) depends only on the *form* of W^1_ν [A]. This is established by the following simple theorems proved in chapter V.

Theorem I : If W^1_ν [A] is of the form

$$W^1_\nu [A] = B \; \epsilon_\nu^{\;\mu\lambda\sigma} A_\mu \partial_\lambda A_\sigma , \tag{6.19}$$

then the result for the family of anomalies of Eqs. (6.6) follows with

$$\beta = -1 + \frac{8\pi^2 i \; B}{e^2} \tag{6.20}$$

provided $W_p[A] = W^o_p[A]$. (These have been defined in Eqs. (6.8) and (6.14) respectively).

For the proof and comments, see chapter V.

Theorem II: If W^1_ν [A] is a local functional of A_μ then W^1_ν[A] is of the form of Eq. (6.19), provided W^1_ν [A] is a "pseudovector" under "parity" transformations [20].

[The "parity" transformations are defined by $\psi \longrightarrow \gamma^4 \psi$; $\bar{\psi} \longrightarrow -\bar{\psi} \gamma^4$ $A_i \rightarrow -A_i$ (i = 1,2,3), $A_4 \longrightarrow A_4$. A'_μ is a "pseudovector" if $A'_i \longrightarrow A'_i$ (i = 1,2,3) and $A'_4 \rightarrow - A'_4$ under "parity" transformations].

In order that W^1_ν [A] is a "pseudovector", it is sufficient that J^{AM}_ν [A] is a "pseudovector". This is guaranteed if X and Y transform as Lorentz scalars, which we shall assume. (see sec. (6.4B)).

It is, thus, evident that if W^1_ν [A] can be shown to be a local functional of A_μ , then even without the need for doing

explicit calculations, the result for the family structure of anomalies would follow. We, thus, proceed to show the locality of W_ν^1 [A] as in chapter V, by examining the form of a typical contribution to it.

Consider, now, W_ν^M [A] defined by Eq. (6.5). We note

$$W_\nu^M [A,x] = - \mathrm{Tr} \int d^4z \, \langle \delta^4(x-z) \, \mathcal{g}(\overleftarrow{Y}) \, \gamma_\nu \gamma_5 \, f(X) \, \psi(x) \, \bar{\psi}(z) \rangle \tag{6.21}$$

Using the definition of the Green's function

$$G(x,z) = - \frac{1}{W[A]} \int D\psi \, D\bar{\psi} \quad e^S \, \psi(x) \, \bar{\psi}(z), \tag{6.22}$$

this becomes

$$W_\nu^M [A,x] = \mathrm{Tr} \int d^4z \, \delta^4(x-z) \, \mathcal{g}(\overleftarrow{Y}) \, \gamma_\nu \gamma_5 \, f(X) \, G(x,z). \tag{6.23}$$

Here, we note that G(x,z;A) has the expansion

$$G(x,z;A) = G_o(x,z) + e \int G_o(x,y) \, \not{A}(y) \, G_o(y,z) \, d^4y + \ldots . \tag{6.24}$$

The expression W_ν^M [A;x] involves

$$\delta^4(x-z) \, \mathcal{g}(\overleftarrow{Y})$$

$$= \frac{1}{(2\pi)^4} \int d^4k \quad e^{ik.(x-z)} \, \mathcal{g}(\overleftarrow{Y}). \tag{6.25}$$

Now $\overleftarrow{Y}$ contains differential operators acting towards left acting on $e^{ik.x}$. We define the right-hand side of Eq. (6.25), after carrying out the differentiation, to be

$$\equiv \frac{1}{(2\pi)^4} \int d^4k \;\; 1.\, q(\overleftarrow{Y}_1) \;\; e^{ik.(x-z)}, \tag{6.26}$$

where $\overleftarrow{Y}_1$ is obtained from $\overleftarrow{Y}$ by replacing $\overleftarrow{\partial}^x_\mu$ by $(\overleftarrow{\partial}^x_\mu + ik_\mu)$ in every place.

In a similar manner, we note that an expansion of Eq. (6.23) using Eq. (6.24) contains in it an expression

$$f(X)\; G_o(x,y) = f(X) \int \frac{d^4k_1}{(2\pi)^4} \; e^{-ik_1.(x-y)} \; \frac{1}{\not{k}_1 - m_o}$$

$$\equiv \int \frac{d^4k_1}{(2\pi)^4} \; e^{-ik_1.(x-y)} \;\; f(X_1) \; \frac{1}{\not{k}_1 - m_o}$$

$$= \int \frac{d^4k_1}{(2\pi)^4} \; e^{-ik_1.(x-y)} \;\; f(X_1) \;.1\; \frac{1}{\not{k}_1 - m_o}, \tag{6.27}$$

where X_1 is obtained from X by replacing the differential <u>operator</u> ∂^x_μ in it (but not derivatives of gauge fields) by $\partial^x_\mu - ik_{1\mu}$.

Noting the definition of X from Eq. (6.9), we obtain

$$X_1 = X_o \left[\frac{(\not{\partial}_x - i\not{k}_1)^2}{M^2} \right] + \tilde{X}' \left[\partial,\; A,\; k_1 \right], \tag{6.28}$$

where the two terms on the right-hand side of Eq. (6.28) are obtained from the respective terms on the right-hand side of Eq. (6.9a) by the replacement $\partial^x_\mu \longrightarrow \partial^x_\mu - ik_{1\mu}$. We re-express

$$X_1 \equiv X_o \left[\frac{-k_1^2}{M^2} \right] + \tilde{X}_o \left[\partial, A, k_1 \right] + \tilde{X}' \left[\partial, A, k_1 \right] \tag{6.29}$$

$$\equiv X_o \left[\frac{-k_1^2}{M^2} \right] + \bar{X}_o \left[\partial, A, k_1 \right] \tag{6.30}$$

We now expand $f(X_1).1$ in Eq. (6.27) in a Taylor series about

$$X_1 = X_o \left[\frac{-k_1^2}{M^2} \right], \text{ as follows:}$$

$$f(X_1).1 = f(X_o[- k_1^2/M^2] + \bar{X}\ [\partial, A, k_1]).1$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)} \left(X_o\ [-k_1^2/M^2] \right) [\bar{X}\ (\partial, A, k_1)]^n.1 \tag{6.31}$$

with $f^{(o)} = f$. A similar expansion will be valid for an analogous quantity $1.g(\overleftarrow{Y}_1)$ appearing in Eq. (6.26) with analogous definitions. These expansions are to be substituted in Eq. (6.23) ultimately.

We now wish to study the form of contributions to W_ν^1 [A], and these are contained in the contributions to the expression (6.23) for W_ν^M [A]. Various contributions to the right hand side of Eq. (6.23) arise when expansions of Eqs. (6.24), (6.31) and a similar expansion for $1.g(\overleftarrow{Y}_1)$ are substituted. For concreteness we take nth-term from the expansion of Eq. (6.24). This term is,

$$e^{n-1} \int \frac{d^4k_1 \ldots d^4k_n}{(2\pi)^{4n}} \frac{1}{\not{k}_1 - m_o} \tilde{\not{A}}\ (k_1 - k_2) \frac{1}{\not{k}_2 - m_o} \tilde{\not{A}}\ (k_2 - k_3) \ldots\ldots$$

$$\ldots\ldots.\tilde{\not{A}}\ (k_{n-1} - k_n) \frac{1}{\not{k}_n - m_o}\, e^{-ik_1.x + ik_n.z} \tag{6.32}$$

A contribution to W_ν^M [A] arises when the above term is substituted in place of G(x,z) in Eq. (6.23) and the qth-term in the expansion of $f(x_1).1$ is taken from Eq. (6.31) and pth-term in the expansion of $1.g(\overleftarrow{\bar{Y}}_1)$ is taken. This contribution upto factors is

$$\mathrm{Tr}\int d^4z \int d^4k\ d^4k_1 \ldots d^4k_n\ e^{ik.(x-z)-ik_1.x+ik_n.z}\ g_o^{(p)}\left(\frac{k^2}{M^2}\right) f_o^{(q)}\left(\frac{k^2}{M^2}\right)$$

$$1.[\overleftarrow{\bar{Y}}\ (\partial,A,k)]^p\ \gamma_\nu\gamma_5\ [\bar{X}\ (\partial,A,k_1)]^q.1$$

$$\times \frac{1}{\not{k}_1-m_o}\ \tilde{\not{A}}\ (k_1-k_2)\ \frac{1}{\not{k}_2-m_o}\ \cdots\cdots\ \tilde{\not{A}}\ (k_{n-1}-k_n)\ \frac{1}{\not{k}_n-m_o}\ , \qquad (6.33)$$

with $0 \le p < \infty$; $0 \le q < \infty$; $1 \le n < \infty$. Here we have introduced the notation $f^{(q)}\left(X_o\ [-k^2/M^2]\right) \equiv f_o^{(q)}\left(\frac{k^2}{M^2}\right)$ and an analogous definition for $g_o^{(p)}$.

We shall now introduce momentum variables

$$q_1 = k-k_1\ ,\quad q_2 = k_1-k_2\ ,\ \ldots\ldots\ ,\quad q_n = k_{n-1} - k_n, \qquad (6.34)$$

so that upto an overall factor, the expression (6.33) becomes

$$\mathrm{Tr}\int d^4z \int d^4q_1\ \ldots\ d^4q_n\ \exp\left\{i\ [q_1.(x-z)\ -\ i(q_2+q_3+\ldots+\ q_n).z]\right\}$$

$$\times \int d^4k\ g_o^{(p)}\left(\frac{k^2}{M^2}\right) f_o^{(q)}\left(\frac{[k-q_1]^2}{M^2}\right)\ 1.[\overleftarrow{\bar{Y}}(\partial,A,k)]^p\gamma_\nu\gamma_5\ [\bar{X}\ (\partial,A,k-q_1)]^q.1$$

$$\times \frac{1}{\not{k}-\not{q}_1-m_o}\tilde{\not{A}}\ (q_2)\ \frac{1}{\not{k}-\not{q}_1-\not{q}_2-m_o}\ \tilde{\not{A}}\ (q_3)\ldots\ldots\tilde{\not{A}}\ (q_n)\ \frac{1}{\not{k}\ -\sum_{i=1}^{n}\not{q}_i-m_o}\ . \qquad (6.35)$$

We now imagine scaling k -> Mk everywhere in the above expression. The relevant k-integral becomes, upto factors,

In a similar manner and with analogous definitions, we would obtain an expression for the other factor $\overleftarrow{\overline{Y}}(\partial.\ A,\ Mk)$ appearing in Eq. (6.36), viz.,

$$\overleftarrow{\overline{Y}}(\partial.\ A,\ Mk) = \left\{ Y_o\left[\frac{[\overleftarrow{\not\partial} - iM\not k\ ]^2}{M^2} \right] - Y_o\left[-\frac{k^2}{M^2} \right] \right\}$$

$$+ \sum_j \frac{\rho_i\ G_j\ [\overleftarrow{\partial} + iMk,\ A,\ \partial A, \ldots.]}{M^{d_j}}\ . \qquad (6.40)$$

Recalling that X_o and Y_o are polynomials in their arguments, the general form of a typical term contributing to the expression (6.36) is

$$M^{4-n}\ \mathrm{Tr} \int d^4k \left(\frac{\not D^2}{M^2}\right)^{\alpha} \left(\frac{k'.D}{M}\right)^{\beta} \prod_i \left[\frac{\lambda_i\ F_i\ [\partial - iMk + iq_1, A, \ldots]}{M^{d_i}}\right]^{\gamma_i} .1\ (k'^2)^{\theta} \gamma_\nu \gamma_5$$

$$\times\ 1. \left[\prod_j \frac{\rho_j G_j\ [\overleftarrow{\partial} + iMk, A, \ldots.]}{M^{d_j}}\right]^{\sigma_j} \left[\frac{\not k'' - m_o/M}{k''^2 - m_o^2/M^2}\right]^n (\not k)^{n-1}\ \varphi_o^{(p)}(k^2)$$

$$\times\ f_o^{(q)}\ ([k - q_1/M]^2), \qquad (6.41)$$

where we have paid no attention to the ordering of γ-matrices, k' refers to either k and/or $k - \frac{q_1}{M}$ amd k'' refers to one of the momenta

$$k - \frac{q_1}{M}\ ,\ \text{or}\ k - \frac{q_1 + q_2}{M}\ ,\ \text{or} \ldots\ldots\ \text{or}\ k - \frac{\sum^n q_i}{M}\ .\quad \text{Here}$$

$$\theta,\ \gamma_i,\ \sigma_j,\ \alpha,\ \beta,\ n-1 = 0,\ 1,\ 2,\ ---- \qquad (6.42)$$

The *leading* power of M in the above expression is seen

to be

$$\Delta_{max} = 4-n-2\alpha-\beta - \sum_i \gamma_i m_i - \sum_j \sigma_j m_j \tag{6.43}$$

We now imagine expanding expression (6.41) in powers of q_i and m_o upto sufficiently many but a finite number of terms. $\left[\right.$ Example: $\frac{1}{x^2-m^2} = \frac{1}{x^2} + \frac{m^2}{x^4} + \frac{m^4}{x^4(x^2-m^2)}\left.\right]$. The actual M power of a typical term, Δ, obtained from such an expression will be less than Δ_{max} by a nonnegative integer η ($\geq$ 0) so that

$$\Delta = 4-n-2\alpha-\beta-\sum_i \gamma_i m_i - \sum_j \sigma_j m_j - \eta \tag{6.44}$$

We, now, look at the infrared behavior of <u>*this*</u> term. Assuming that $q_o^{(p)}(0)$ and $f_o^{(q)}(0)$ are finite, this term, in the infrared region, behaves as $k^{\Delta'}$ (counting d^4k) where

$$\Delta' = 4+\beta-n + \sum_i l_i \gamma_i + \sum_j l_j' \sigma_j - \eta' + 2\theta , \tag{6.45}$$

where η' ($\geq$ 0) is an integer. An inspection of Eq. (6.41) will convince the reader that

$$\eta - \eta' \geq 0 . \tag{6.46}$$

Uptil now, we have been considering a general term contributing to W_ν^M [A], which contains all the terms contributing to $W_\nu^1[A]$, in which we are really interested. Now a term contributes to

$$\begin{aligned} W_\nu^1 &= W_\nu - W_\nu^o \\ &= W_\nu - W_\nu \Big|_{\lambda_i=0,\ \rho_j=0} , \end{aligned} \tag{6.47}$$

(See Eqs (6.9) and (6.37)) only if power of some λ_i and/or ρ_j is positive. This is so if (γ and σ are integers)

$$\sum_i \gamma_i + \sum_j \sigma_j \geq 1 . \tag{6.48}$$

Hence as all $m_i \geq 1$, Eq. (6.48) is valid iff

$$\sum_i \gamma_i m_i + \sum_j \sigma_j m_j \geq 1 \tag{6.49}$$

We, now, focus our attention on terms that contribute to W_ν^1 [A] as $M \longrightarrow \infty$. For such a term

$$\Delta \geq 0 \tag{6.50}$$

We then find that for *such* a term

$$\begin{aligned} \Delta' &= 4+\beta-n + \sum_i l_i \gamma_i + \sum_j l'_j \sigma_j - \eta' + 2\theta \\ &= \Delta + (2\alpha + 2\beta) + \sum_i \gamma_i m_i + \sum_j \sigma_j m_j + (\eta-\eta') \\ &\quad + \sum l_i \gamma_i + \sum_j l'_j \sigma_j + 2\theta \\ &\geq 1 \end{aligned} \tag{6.51}$$

on account of Eqs (6.50), (6.42), (6.49), (6.46) and the fact that $l_i, \gamma_i, \sigma_j \geq 0$. $\Delta' \geq 1$ implies that the k-integral under consideration is infrared convergent.

The result of Eq. (6.51) is what we have been seeking in this section. As seen below, this guarantees the locality of W_ν^1 [A]. This can be formulated as below:

Theorem III : If $\Delta' \geq 1$ for each term contributing to W_{ν}^{1} [A] from the right hand side of Eq. (6.35), then W_{ν}^{1} [A] is local provided the k-integrals involved are ultraviolet convergent.

The proof of the theorem proceeds exactly, verbatim, as in chapter V. Hence, it is not reproduced here. [The conditions under which the k-integrals are ultraviolet convergent are to be discussed in detail in Sec. (6.4)]. Given the result proved in theorem III about the locality of W_{ν}^{1} [A], theorem II implies that W_{ν}^{1} [A] has the form of Eq. (6.19). From theorem I, then the family structure of anomalies follows, provided $W_p[A] = W_p^o[A]$. The parameter β in the family structure depends on B in Eq. (4.19) which in turn depends on X, Y, f and q in a complicated manner.

Now we shall give an argument to show that $W_p[A] = W_p^o[A]$. A similar statement has been proved in a restricted context $X = Y = \dfrac{\not{\mathcal{D}}_a^2}{M^2}$ in chapter V. This proof essentially is based on an argument uses the machinery already developed in this section. From Eq. (6.8) we have

$$W_p[A] = \lim_{M \longrightarrow \infty} \langle \bar{\psi}(x)\ q(\overleftarrow{Y}_o + \overleftarrow{Y}')\gamma_5\ f(X_o + X')\psi(x)\rangle \qquad (6.52)$$

while from Eqs. (6.14), (6.12) and (6.13) it is seen that

$$W_p^o[A] = \lim_{M \longrightarrow \infty} \langle \bar{\psi}(x)\ q(\overleftarrow{Y}_o)\gamma_5\ f(X_o)\psi(x)\rangle \qquad (6.53)$$

is obtained from $W_p[A]$ by setting the contributions from Y′ and X′ parts of operators to zero. We separate this contribution as $W_p^1[A]$ where

$$W_p[A] = W_p^o[A] + W_p^1[A] \tag{6.54}$$

The treatment of terms contributing to $W_p^1[A]$ is exactly similar to that of $W_\nu^1[A]$ in this section. Consequently a similar argument shows that $W_p^1[A]$ is local under similar conditions on f, q, X, Y. But $W_p^1[A]$ is a "pseudoscalar" function of A of dimension three which does not exist. Hence $W_p^1[A] = 0$.

6.4 CONSTRAINTS ON OPERATORS X, Y AND FUNCTIONS f AND q

In the previous section we presented the essential argument that lead to the derivation of the family of anomalies in a very general context. In the process of proving the result we made a number of assumptions on the opertors X, Y and functions f and q that are necessary for the results of the previous section to hold. For example it was assumed that the operators and functions are such that J_ν^{AM} is a "pseudo vector". It was assumed that all the integrals involved in Eq. (6.41) are ultraviolet convergent. These do place restrictions on X, Y, f, q. The purpose of this section is to explore these "loose ends" in the previous section and precisely formulate conditions on X, Y and functions f, q that will make the proof of previous section completely rigorous.

A. Constraints on functions f and q

Our aim is to formulate sufficient conditions for the integrals in Eq. (6.41) to be ultraviolet convergent. For this purpose, it is useful to characterise operator X by two indices. Referring to Eq. (6.9) let X be a polynomial in $\frac{\not{D}^2}{M^2}$ of degree N_1.

Referring to Eq. (6.37), let the maximum number of derivative operators amongst all of $F_i[\partial, A, \partial A, ..]$ by N_2. i.e. $N_2 = \max\{l_i\}$. Then X will be characterized by the pair of integers (N_1, N_2). In a similar manner let Y be characterized by $(\bar{N}_1, \bar{N}_2)$.

We now examine the ultraviolet behavior of the integrand in Eq.(6.41). The worst behavior as $k^2 \longrightarrow \infty$ is of the form

$$k^{(\beta - n + 2\theta + \sum_i l_i \gamma_i + \sum_j l'_j \sigma_j)} g_o^{(p)}(k^2) f_o^{(q)}(k^2) \tag{6.55}$$

Here, $\beta, \theta, \gamma_i \sigma_j$ are not completely arbitrary but restricted given p and q as seen below. We note that contribution of Eq. (6.41) has arisen out of $[\bar{Y}]^p\ [\bar{X}]^q$. $\bar{X}$ has two contributions as seen from Eq. (6.30) : $\bar{X} = \tilde{X}_o + \tilde{X}'$. The term of Eq. (6.41) under consideration may have arisen out of $[\tilde{X}_o]^{\beta_1}[\tilde{X}']^{q-\beta_1}$ type term in the expansion of $[\bar{X}]^q$. Then evidently, noting the form of X_1 from Eq.(6.37),

$$\beta_1 + \sum \gamma_i = q \ . \tag{6.56}$$

In a similar manner and with analogous definitions

$$\bar{\beta}_1 + \sum_j \sigma_j = p \ . \tag{6.57}$$

The power of k in Eq.(6.55) arising out of $[\tilde{X}_o]^{\beta_1}\ [\tilde{Y}_o]^{\bar{\beta}_1}$ type term is $k^{2\theta+\beta}$. The maximum power of k in $[\tilde{X}_o]$ is k^{2N_1-1}. Hence the maximum power of k in $[\tilde{X}_o]^{\beta_1}\ [\tilde{Y}_o]^{\bar{\beta}_1}$ type term is $k^{2\beta_1 N_1 + 2\bar{\beta}_1 \bar{N}_1 - (\beta_1 + \bar{\beta}_1)}$ and occurs when $\alpha = 0$. When α is not zero, this maximum power is reduced at least by α. Hence,

$$2\theta + \beta \leq 2\beta_1 N_1 + 2\bar{\beta}_1 \bar{N}_1 - \beta_1 - \bar{\beta}_1 - \alpha \cdot \qquad (6.58)$$

Now we wish to put an upper bound on the power of k appearing in the expression (6.55) in terms of p,q. Consider

$$\beta + 2\theta + \sum_i l_i \gamma_i + \sum l'_j \sigma_j - n$$

$$\leq 2\beta_1 N_1 + 2\bar{\beta}_1 \bar{N}_1 - (\beta_1 + \bar{\beta}_1) - \alpha + \sum_i l_i \gamma_i + \sum l'_j \sigma_j - n$$

$$\leq 2\beta_1 N_1 + 2\bar{\beta}_1 \bar{N}_1 - (\beta_1 + \bar{\beta}_2) - \alpha + (\sum \gamma_i) N_2 + (\sum \sigma_j) \bar{N}_2 - n$$

$$\leq (\beta_1 + \sum \gamma_i) N + (\bar{\beta}_1 \sum \sigma_j) \bar{N} - (\alpha + n)$$

$$\leq Nq + \bar{N} p - (\alpha + n), \qquad (6.59)$$

where we have set $N = \max\{2N_1 - 1, N_2\}$ and $\bar{N} = \max\{2\bar{N}_1 - 1, \bar{N}_2\}$, and have used Eqs. (6.58), (6.56) and (6.57) successively.

To discuss the ultraviolet behavior of (6.55), we need to consider that of $\varphi_o^{(p)}(k^2)$ and $f_o^{(q)}(k^2)$. First we note that

$$X_o[-k^2] = O(k^{2N_1}) \quad \text{as} \quad k^2 \longrightarrow \infty \qquad (6.60)$$

Now, if we assume that

$$f^{(q)}(x) = O(x^{b(q)}) \quad \text{as} \quad |X| \longrightarrow \infty \qquad (6.61)$$

then

$$f_o^{(q)}(k^2) = f^{(q)}[X_o(-k^2)] = O[k^{2N_1 b}] \text{ as } k^2 \longrightarrow \infty \qquad (6.62)$$

In a similar manner and with analogous definitions

$$q_o^{(p)}(k^2) = O\,[k^{2\bar{N}_1\bar{b}}] \text{ as } k^2 \longrightarrow \infty \,. \tag{6.63}$$

Thus the expression (6.55) is

$$O\left(k^{\beta+2\theta} + \sum l_i\gamma_i + \sum l_j' \sigma_j - n + 2N_1 b + 2\bar{N}_1\bar{b}\right) \text{ as } k^2 \longrightarrow \infty.$$

Hence the ultraviolet convergence of the integral in Eq. (6.41) is assured if, using Eq. (6.59),

$$Nq + \bar{N}p + 2N_1 b(q) + 2\bar{N}_1\,\bar{b}(p) < -4 + (\alpha+n). \tag{6.64}$$

$$p,\ q = 0,\ 1,\ 2,\ \ldots\ldots.$$

The above is a set of sufficient conditions on the pair of functions f and g, given $N_1,\ N_2,\ \bar{N}_1,\ \bar{N}_2$.

The conditions of Eq. (6.64) are not too illuminating as they stand being infinite in number. They can, however, be simplified under certain restrictive assumptions of f and g.

Firstly, in view of the fact that $n \geq 1$ amd $\alpha \geq 0$, Eq. (6.64) is satisfied if

$$Nq + \bar{N}p + 2N_1\, b(q) + 2\bar{N}_1\,\bar{b}(p) < -3\ . \tag{6.65}$$

Now suppose we restrict ourselves to the set (still a wide class) of functions with

$$\begin{aligned} f(x) &= O(x^c) \quad \text{and} \quad f^{(q)}(x) = O(x^{c-q}) \\ g(x) &= O(x^d) \quad \text{and} \quad g^{(p)}(x) = O(x^{d-p}) \\ &\qquad p,\ q = 1,2,\ \ldots\ldots \end{aligned} \tag{6.66}$$

as $|x| \to \infty$ and $c,d \leq 0$, then

$$b(q) = c-q\ , \qquad \bar{b}(p) = d-p\ , \tag{6.67}$$

and one has

$$2N_1C + 2\bar{N}_1d + p(\bar{N} - 2\bar{N}_1) + q(N - 2N_1) < -3 \tag{6.68}$$

$$q,\ p = 0,\ 1,\ 2,\ \ldots\ .$$

Now we consider two separate cases:

Case I : $\bar{N} - 2\bar{N}_1 \le 0$, $N - 2N_1 \le 0$ (This happens if $N_2 \le 2N_1$, $\bar{N}_2 \le 2\bar{N}_1$). In this case Eq. (6.68) is guaranteed for all $p,q \ge 0$ if

$$2N_1C + 2\bar{N}_1d < -3 \tag{6.69}$$

and thus a single sufficient condition on f and g is formulated.

Case II : Either $\bar{N} - 2\bar{N}_1 > 0$ and/or $N-2N_1 > 0$. (This happens if $\bar{N}_2 \ge 2\bar{N}_1 + 1$ and/or $N_2 \ge 2N_1 + 1$) . In this case Eqs. (6.68) are satisfied for all $p,\ q \ge 0$ iff

$$2N_1C + 2\bar{N}_1d \longrightarrow -\infty \tag{6.70}$$

which, in view of $N_1, \bar{N}_1 > 0$ requires either $c \longrightarrow -\infty$ and/or $d \longrightarrow -\infty$. Such functions belong to what are called in mathematical literature the "Schwarz class".

Next we give some examples.

(i) $X = Y = \dfrac{\not{p}^2}{M^2}$. Here $N_1 = \bar{N}_1 = 1$. $N_2 = \bar{N}_2 = 0$. (6.71a)

$$f(x) = \frac{1}{Ax+1} \qquad g(x) = \frac{1}{Bx+1} \qquad A,B \ne 0. \tag{6.71b}$$

Then $c = d = -1$. And

$$2N_1c + 2\bar{N}_1 d = -4 < -3 \tag{6.72}$$

so that Eq. (6.69) is satisfied.

(ii) $f(x) = e^{-x}$. Hence $c \longrightarrow -\infty$. Thus both Eqs. (6.69) and (6.70) are satisfied. Hence $g(X)$, X, Y can be anything subject to constraints $g(0) = 1$, $d \leq 0$, and constraints on X and Y to be outlined later.

The sufficient conditions can be further simplified in some simpler cases.

Case I : $f = g$ $\quad X = Y$

Eq. (6.69) reduces to $4N_1c < -3$, so that

$$c < -\frac{3}{4N_1} \tag{6.73}$$

Case II : $g = 1$

In this case $g^{(p)} = 0 \quad p \geq 1$. $\bar{b}(0) = 0$. In this case the discussion proceeds as in the general case except that terms depending on σ_j, p, $\bar{N}_1$, $\bar{N}_2$, $\bar{N}$ simply do not arise. Hence Eq. (6.65) simplifies to

$$Nq + 2N_1\, b(q) < -3 \tag{6.74}$$

$$q = 0, 1, 2, \ldots\ldots$$

under the assumptions of Eq. (6.66), $b(q) = c-q$. Hence we have

$$(N - 2N_1)q + 2N_1 c < -3. \tag{6.75}$$

If $N \leq 2N_1$, this requires that

$$c < -3/2N_1 \tag{6.76}$$

If $N > 2N_1$, this requires that $c \longrightarrow -\infty$, i.e. f belongs to the "Schwarz class". The case with $f = 1$ can be dealt with symmetrically.

Finally, we shall make a comment. The condition of Eq. (6.65) which we have derived is a *sufficient* condition and not a necessary condition. In some cases, even a weaker condition proves sufficient; a useful example is being given now. Consider the case $q = 1$, $f = \frac{1}{x+1}$; $X = \frac{\not{D}^2}{M^2}$. These functions do not fulfil the condition of the equation (6.69); (for, here, $d = 0$, $c = -1$, $N_1 = 1$, $N_2 = 0$). Even then, these functions and the operator X can be used to regularize the current, this being a Pauli-Villars type regularization. This happens because an inspection of Eq. (6.41) will show that due to the insufficient number of γ matrices the case $\alpha = 0$, $n = 1$ does not contribute and hence $\alpha+n \geq 2$ always. Also for $q = 0 = p$, the terms contribute only from $n \geq 3$. As a result of this, the conditions for convergence are actually fulfilled by this choice of functions and operators, even though the sufficient conditions of Eq. (6.69) are not fulfilled.

B. Conditions on operators X and Y

For technical reasons, we have chosen the operator X such that at $A = 0$,

$$X = X_o \left[\frac{\not{\partial}^2}{M^2} \right]$$

We have expressed [See Eq (6.9a)]

$$X = X_o \left[\not{D}^2/M^2\right] + X'\left[\partial, A, \partial A \ldots .\right]$$

$$X = X_o \left[\frac{\not{\partial}^2}{M^2}\right] + X'' \left[\partial,\ A,\ \partial A,\ \ldots .\right] \qquad (6.9a)$$

where X" vanishes at A = 0. We shall, if necessary, treat gauge field A as a perturbation. Hence X" will be treated as a perturbation over X_o [in the sense of their eigenvalues]. X_o is hermitian and has a complete set of eigenfunctions. Eignevalues of X are infinitesimally different from these of X_o. Since the regularizing functions f and q are so chosen that they will provide damping for high eigenvalues of X_o, they do so for eigenvalues of X which are large (or have a large real part). A complete set of eigenfunctions of X can be constructed perturbatively from those of X_o.

To obtain restrictions of X and Y we note that we need the expression for the regularized current of Eq. (6.3)

$$J_\mu^{AM} = \bar{\psi}(x)\ q(\overleftarrow{Y})\ \gamma_\mu \gamma_5 f(X)\ \psi(x) \qquad (6.3)$$

to be a "pseudovector". [See Theorem II and paragraph below it in Sec. 6.3]. This is guaranteed if (and probably only if)

$$S^{-1}\ f(X)\ S = f(X)$$

and

$$S^{-1}\ q(\overleftarrow{Y}) S = q(\overleftarrow{Y}) \qquad (6.77)$$

[where S is defined as the SO(4) rotation $\psi \longrightarrow S\psi$ and $\bar{\psi} \longrightarrow \bar{\psi}S^{-1}$]. As f and g are assumed to be Taylor series in X and Y, Eq.(6.77) is guaranteed if

$$S^{-1}\, XS = X, \quad S^{-1}\, YS = Y \ . \tag{6.78}$$

That is, the operators X and Y are SO(4) scalars. X" and Y"can be non hermitian, but scalars.

We now discuss further, the restrictions we have already placed on X and Y, viz., X and Y are local operators and do not contain additional dimensionful parameters. We cannot relax these restrictions in a general sense. To see this we give some counter examples

(i) Non-local operators *can* generate non-local contributions to $W_\nu^{\ 1}[A]$, so their use will lead to a breakdown of our arguments, which are based on the locality of $W_\nu^{\ 1}[A]$. For example, consider

$$X = \frac{\partial\!\!\!/^2}{M^2} + \frac{1}{M^2}\left[\frac{1}{\partial^2}\,\partial.A\right]\sigma_{\mu\nu}\ F^{\mu\nu}.$$

Such forms of X are not allowed; for this generates non-local contributions to $W_\nu^{\ 1}[A]$ of the form $\left[\frac{1}{\partial^2}\,\partial.A\right]\varepsilon_{\nu\mu\lambda\sigma}\ A^\mu\ F^{\lambda\sigma}$.

(ii) X must also not generally contain additional dimensionful parameters, e.g., the choice $X = \frac{\partial^2}{M^2} + \left(\frac{\partial.A}{m_o^{\ 2}}\right)\frac{\partial^2}{M^2}$ will generate higher dimension contributions to $W_\nu[A]$, e.g., $\partial.A\ \varepsilon_\nu^{\ \lambda\rho\sigma}\ A_\lambda\ \partial_\rho\ A_\sigma$ etc.

6.5 ANOMALY AND JACOBIAN CONTRIBUTION

In this section, we wish to generalize the standard procedure of Fujikawa for obtaining chiral anomaly via the field transformations and the consequent Ward Identities in which a term will be interpreted as the regularized jacobian. In Fujikawa's treatment which uses the operator $\not{D}$, the jacobian term is the only nontrivial term. In our general treatment there will be additional terms which can be interpreted as arising from action S[13]. The treatment given here is more general in two senses: (i) Our choice of operator X for defining path integral measure is very general, (ii) Our treatment is well defined at every stage and no *adhoc* regularization is introduced in an intermediate stage; unlike the treatment in chapter II. This is accomplished by making use of Ward Identities associated with a modified form of infinitesimal chiral transformations. (Note that *any* transforma tion of integration variables leads to some W-T-like identity). The W-T identity in this case has well defined regularized terms. In particular, the anomalous jacobian term ~ $\Sigma\, \phi_n^{\dagger}\, \gamma_5\, \phi_n$ arises here in a regularized form automatically (see Eqns (6.86) and (6.89) below).

We proceed as follows. We first derive the required W-T identity and then derive the anomaly equation by computing the axial-vector current divergence for the current defined in Eq. (6.3) by setting X = Y.

A. W-T Identities

We expand the fermion fields ψ and $\bar{\psi}$ in terms of the complete set eigenfunction of the hermitian operator X of the kind given by Eq. (6.4) :

$$X\,\phi_n = \lambda_n\,\phi_n\,,\qquad \phi_n^{\dagger}\,\overleftarrow{X} = \lambda_n\,\phi_n^{\dagger}\,. \tag{6.79}$$

$$\psi = \sum_n a_n\,\phi_n\,,\qquad \bar{\psi} = \sum_n \phi_n^{\dagger}\,\bar{b}_n\,. \tag{6.80}$$

$$\int \phi_n^{\dagger}(x)\,\phi_m(x)\,d^4x = \delta_{nm}\,.$$

We then define

$$W[A] = \int \prod_n da\,\Pi_n d\bar{b}_n\,e^{\sum_{nm} \bar{b}_n\,a_m\,\xi_{nm}} \equiv \int DaD\bar{b}\,e^{S[a\,\bar{b}]}, \tag{6.81}$$

where $\xi_{nm} \equiv \int d^4x\,\phi_n^{\dagger}\,(i\,\not{\partial} - m_o)\,\phi_m\,.$ (6.82)

We now define "modified infinitesimal chiral transformations" with x-dependent parameter $\alpha(x)$ by

$$\psi'(x) = \sum_n a'_n\,\phi_n(x) = \psi(x) + i g(X)\,\alpha(x)\gamma_5 f(X)\psi(x) \tag{6.83}$$

$$\bar{\psi}'(x) = \sum_n \bar{b}'_n\,\phi_n(x) = \bar{\psi}(x) + i\bar{\psi}(x)\,g(\overleftarrow{X})\alpha(x)\gamma_5 f(\overleftarrow{X}) \tag{6.84}$$

Thus

$$a'_n = \int \varphi_n^{\dagger}\,(x)\,\psi'(x)\,d^4x$$

$$= \int \varphi_n^{\dagger}\,(x)\,\psi(x)\,d^4x + i\int \varphi_n^{\dagger}\,g(X)\,\alpha(x)\gamma_5\,f(X)\,\psi(x)\,d^4x$$

$$= a_n + i\int \varphi_n^{\dagger}\,g(\overleftarrow{X})\,\alpha(x)\gamma_5\,f(X)\,\sum_m a_m\phi_m(x)\,d^4x$$

$$= a_n + i\, g(\lambda_n) \sum_m a_m \int d^4x\, \varphi_n^\dagger(x)\, \alpha(x)\gamma_5\, \varphi_m(x)\, f(\lambda_m)$$

$$= a_n + \sum_m C_{nm}\, a_m \, , \tag{6.85}$$

with

$$C_{nm} = g(\lambda_n)\, f(\lambda_m)\, i \int d^4x\, \varphi_n^\dagger\,(x)\, \alpha(x)\, \gamma_5\, \varphi_m(x) \tag{6.86}$$

In a similar manner one obtains, from Eq. (6.84),

$$\bar{b}'_n = \bar{b}_n + \sum C_{mn}\, \bar{b}_m \, . \tag{6.87}$$

These transformations lead to the jacobian $J(\alpha)$:

$$D\bar{b}\, Da = J(\alpha)\, D\bar{b}'\, Da', \tag{6.88}$$

with [7]

$$\ln J(\alpha) = 2 \sum_n C_{nn} \, . \tag{6.89}$$

The action $S[a,\bar{b}]$, when expressed in terms of new variables, becomes,

$$S[a,\bar{b}] \equiv S'[a',\bar{b}'] \equiv S\,[a',\bar{b}'] + \Delta S\,[a',\bar{b}'], \tag{6.90}$$

with

$$\Delta S[a',\bar{b}'] = - \sum_p \sum_q \sum_n \bar{b}'_p\, a'_q\, (C_{pn}\, \xi_{nq} + C_{nq}\, \xi_{pn}) . \tag{6.91}$$

Now equating the two forms of $W[A]$

$$W[A] = \int D\bar{b}\, Da\;\; e^{S[a,\bar{b}]}$$

$$= \int D\bar{b}'\, Da'\;\; e^{S[a',\bar{b}'] + \Delta S\,[a',\bar{b}'] + \ln J(\alpha)}, \tag{6.92}$$

leads us to the W-T identity

$$\int D\bar{b}\, Da\, e^{S[a\,\bar{b}\,]} \{ \Delta S\,[a,\bar{b}] + \ln J(\alpha) \} = 0. \tag{6.93}$$

From Eqs. (6.89) and (6.91), this becomes

$$\langle \sum_p \sum_q \sum_n \bar{b}_p\, a_q\, (C_{pn}\, \xi_{nq} + C_{nq}\, \xi_{pn}) - 2 \sum_n C_{nn} \rangle = 0, \tag{6.94}$$

where

$$\langle\, 0\, \rangle = \frac{1}{W\,[A]} \int D\bar{b}\, Da\, e^{S}\, 0\ . \tag{6.95}$$

Eq. (6.94) is the *regularized* version of the chiral W-T identity of Eq. (2.20) of chapter II, [because C_{pn} now contains regulator functions $g(\lambda_p)\, f(\lambda_n)$]. It has been obtained from our "modified infinitesimal chiral transformations" of Eqs.(6.83) and (6.84).

B. Anomaly Equation

We start from the definition of $J^{AM}_{\mu}(x)$ of Eq. (6.3), setting X = Y.

$$\begin{aligned} J^{AM}_{\mu}(x) &= \bar{\psi}(x)\, g(\overleftarrow{X})\, \gamma_\mu\, \gamma_5\, f(X)\, \psi(x) \\ &= \sum_n \sum_m b_n\, a_m\, g(\lambda_n)\, f(\lambda_m)\, \phi^{\dagger}_n(x)\, \gamma_\mu \gamma_5\, \phi_m(x), \end{aligned} \tag{6.96}$$

where λ_n and φ_n are defined by Eq. (6.79). We then compute

$$\begin{aligned} \partial^\mu J^{AM}_{\mu}(x) = -\sum_n \sum_m \bar{b}_n\, a_m\, g(\lambda_n)\, f(\lambda_m) \Big\{ &\varphi^{\dagger}_n(x)\, \gamma_5 [\not{\partial} + im_o] \varphi_m(x) \\ &+ \varphi^{\dagger}_n(x)\, [\overleftarrow{\not{\partial}} + im_o] \gamma_5\, \varphi_m(x) \Big\} \\ + 2im_o \sum_n \sum_m \bar{b}_n\, a_m\, g(\lambda_n)\, f(\lambda_m)\, &\varphi^{\dagger}_n(x)\, \gamma_5\, \varphi_m(x). \end{aligned} \tag{6.97}$$

We have defined the last term in Eq.(6.97) to be $+2im_o\ W_p^M[A]$. Then using Eq. (6.94) in the above equation

$$\int d^4x\ \alpha(x)\ \langle \partial^\mu\ J_\mu^{AM}(x)\ -2im_o\ (\bar\psi\ \gamma_5\ \psi)^M \rangle$$

$$= 2\sum_n c_{nn} - \Big\langle \sum_n \sum_m \bar{b}_n\ a_m \Big\{ \sum_p (c_{np}\ \xi_{pm} + c_{pm}\ \xi_{np})$$

$$+ \int d^4x\ \alpha(x)\ f(\lambda_m)\ g(\lambda_n) \Big[\varphi_n^\dagger\ \gamma_5\ [\not{D} + im_o]\varphi_m$$

$$+ \varphi_n^\dagger\ (x)\ [\overleftarrow{\not{D}} + im_o]\gamma_5\varphi_m\Big]\Big\} \Big\rangle \tag{6.98}$$

The right-hand side of Eq. (6.98) is the chiral anomaly. The first term is the regularized jacobian. The second term is the extra contribution which may be interpreted as arising from the action [13]. We shall show that when X is of the form $X = X_o[\not{D}^2/M^2]$, this second term vanishes so that it receives contributions only from $X - X_o\ [\not{D}^2/M^2]$ in the general case. This also explains why jacobian term alone is sufficient in the Fujikawa's derivation.

To see this consider X of the form, $X = X_o[\not{D}^2/M^2]$ As in Eq. (6.79) we expand ψ in terms of eigenfunctions of X_o, which can be chosen possibly in more than one way. The result for the right hand side of Eq. (6.98) cannot depend on the specific choice for the set as different choices of bases are related by a unitary transformation. In particular eigenfunctions of $\not{D}$ are also

eigenfunctions of $X_o\left[\frac{\not{D}^2}{M^2}\right]$ and form a complete set. Let us choose φ_n to be these eigenfunctions of $\not{D}$:

$$\not{D}\,\varphi_n = \lambda'_n\,\varphi_n\,.$$

$$X_o\,\varphi_n = X_o\left[\frac{\lambda'_n}{M^2}\right]\varphi_n\,. \tag{6.99}$$

Now, in this basis

$$\langle\,\bar{b}_n\,a_m\,\rangle = \frac{\delta_{mn}}{i\,\lambda'_n - m_o}\,. \tag{6.100}$$

Thus the second term on the right-hand side of Eq. (6.98) becomes

$$-\left[\sum_n\sum_p \frac{2C_{np}\,\xi_{pn}}{i\lambda'_n - m_o} + 2\sum_n f(\lambda_n)\,g(\lambda_n)\int d^4x\,\alpha(x)\varphi_n^\dagger(x)\gamma_5\,\varphi_n(x)(-i)\right] \tag{6.101}$$

Noting that from Eqs. (6.82) and (6.99)

$$\xi_{pn} = (\,i\lambda'_p - m_o)\,\delta_{pn} \tag{6.102}$$

and from Eq. (6.86)

$$C_{nn} = g(\lambda_n)\,f(\lambda_n)\,i\int d^4x\,\varphi_n^\dagger(x)\,\alpha(x)\,\gamma_5\,\varphi_m(x) \tag{6.103}$$

the expression (6.101) is seen to vanish proving the result mentioned earlier. This shows that the second term contributes only when X differs from the form $X_o\,[\not{D}^2/M^2]$ and is responsible for the freedom in the family of anomaly equations.

We have seen in the discussion above that in our general

formulation of anomalies, involving derivation of anomaly equations which is (a) *well-defined* at each step and (b) which involves use of general bases (which can differ from the "$\not{D}$ basis"), the contribution to the anomaly comes not only from the jacobian but from additional terms which can be interpreted as coming from the action.

We present below, an example which shows in a simple way that identification of anomaly *a priori* with jacobians is not correct. (Such identifications are made by formal derivations using unregularized quantities and are therefore suspect). If on the other hand, we deal with regulated quantities, anomaly will have contribution other than jacobian. (For example, see Eq. (6.107) below).

Suppose we want the vector anomaly equation in $(QED)_2$ involving $\langle\partial^\mu J^V_\mu\rangle$.

To derive the equation for $\langle\partial^\mu J^V_\mu\rangle$, consider the *regulated* path-integral

$$W^R[A] = \int D\psi D\bar{\psi}\, e^{\int d^2x\, [\bar{\psi}(i\not{\partial} - e_o e^{-\not{D}^2/M^2} \not{A}\, e^{-\not{\partial}^2/M^2})\psi]} \tag{6.104}$$

$$\equiv \int D\psi D\bar{\psi}\, e^{S^R} .$$

Here the regularized action $S^R = \int d^2x\, (\bar{\psi}(i\not{\partial} - e_o e^{-\not{D}^2/M^2} \not{A}\, e^{-\not{\partial}^2/M^2})\psi)$ is clearly not gauge invariant.

Under gauge transformations we will have

$$- \int \alpha(x)\ d^2x\left(i\bar{\psi}\ \frac{\delta S^R}{\delta\bar{\psi}} - i\ \frac{\delta S^R}{\delta\psi}\psi - \frac{1}{e}\ \partial^\mu\ \frac{\delta S^R}{\delta A_\mu} \right) = \Delta S^R\ .$$

Consider the change of integration variables in $W^R[A]$ given by the local transformations

$$\begin{aligned} \psi' &= [\ 1 + i\alpha(x)\ ]\ \psi \\ \bar{\psi}' &= \bar{\psi}\ [\ 1 - i\alpha(x)\ ]\ . \end{aligned} \tag{6.105}$$

In terms of the new variables we have

$$W^R[A] = \int D\psi'\ D\bar{\psi}'\ e^{\ln J\ +\ S^R\ +\ \Delta S^R\ -\ \frac{1}{e}\int d^2x\alpha(x)\,(\partial^\mu\ \frac{\delta S^R}{\delta A_\mu})}\ . \tag{6.106}$$

Define

$$J_\mu^{\ VR} \equiv - \frac{1}{e}\ \frac{\delta S^R}{\delta A_\mu} = \bar{\psi}\ e^{-\overleftarrow{\not{\partial}}^2/M^2}\ \gamma_\mu\ e^{-\not{\partial}^2/M^2}\ \psi\ .$$

Equating the two forms of $W^R[A]$ given in equations (6.104) and (6.106) we get

$$\mathop{\mathrm{Lt}}_{M \longrightarrow \infty} \left[\int d^2x\alpha(x) \langle \partial^\mu J_\mu^{\ VR} \rangle = \mathop{\mathrm{Lt}}_{M \longrightarrow \infty} \left[\ln J + \langle - \Delta S^R \rangle \right] . \right. \tag{6.107}$$

Clearly there is an extra anomalous term (the last one) which would have been missed had limit $M \longrightarrow \infty$ been taken initially in W^R, i.e., if one had used unregularized path-integral, W. [Of course, here ln J happens to vanish.]

Even when gauge-covariance is maintained, by making use of gauge-covariant operators to define the path integral measure

and regularized current, it is not difficult to see that there can be contributions from sources other than "jacobian sources". We see this in a simple way as follows.

In two dimensions, suppose we define our regularized currents as

$$J_\mu^{AM} = \bar{\psi}(x)\, e^{-\frac{\overleftarrow{X}}{M^2}}\, \gamma_\mu \gamma_5\, e^{-\frac{X}{M^2}}\, \psi(x) \tag{6.108a}$$

$$J_\mu^{VM} = \bar{\psi}(x)\, e^{-\frac{\overleftarrow{X}}{M^2}}\, \gamma_\mu\, e^{-\frac{X}{M^2}}\, \psi(x) \tag{6.108b}$$

where $X = \not{D}^2 + \tilde{a}\, \varepsilon_{\mu\nu}\gamma_5 F^{\mu\nu}$, $\tilde{a}$ being a free parameter. Regularization is covariant. Hence, we expect no anomaly in the vector current. Therefore, in the family of anomaly equations (see equations (3.18)), we expect the chiral anomaly equation with a = 1. As we have seen, the chiral anomaly equation with a=1 is obtained if X is chosen to be $\not{D}^2$ and in this case chiral anomaly is entirely from the jacobian. But in the present case, with $X = \not{D}^2 + \tilde{a}\, \varepsilon_{\mu\nu}\gamma_5 F^{\mu\nu}$, there is an additional contribution to chiral jacobian proportional to $\tilde{a}$, as is easily seen. Thus the chiral anomaly equation with a=1 would be obtained as expected, only if there is an additional contribution to chiral anomaly, cancelling this extra piece in jacobian contribution, which indeed is the case. This example clearly indicates that there is an additional contribution to the anomaly which is different from the jacobian contribution, even for a gauge covariant regularization.

6.6 KNOWN REGULARIZATIONS AS SPECIAL CASES

The purpose of this section is to show that a number of known regularizations of triangle diagram are special cases of our discussion. In this section we shall consider (1) Fujikawa's derivation [7] (2) Family of anomalies derivation of chapter V [20] (3) Feynman diagrammatic method with exponential regularization (4) A Feynman diagrammatic method with Pauli Villars type regularization (5) Proper time method. Various other regularization schemes using operators different from $\not{D}^2/M^2$ and operators different for expansion of ψ and $\bar{\psi}$ have also been used [17,26]. Such treatments applied to QED are seen to be special cases of this work also.

(1) Fujikawa's derivation of anomaly

We set $X = \dfrac{\not{D}^2}{M^2} = Y$, $f(X) = e^{-X}$, $g(Y) \equiv 1$.

Here $W_\nu[A] = W_\nu^o[A]$.

Using Eq.(6.15),

$$\partial^\nu W_\nu^{oM} = 2im_o\, W_p^{oM}[A]$$

$$- \langle \bar{\psi}\, [\overleftarrow{\not{D}}+im_o]\gamma_5\, e^{-\not{D}^2/M^2}\, \psi(x)\rangle - \langle \bar{\psi}(x)\gamma_5\, e^{-\not{D}^2/M^2} (\not{D}+im_o)\psi(x)\rangle \tag{6.109}$$

Consider the last term

$$- \langle \bar{\psi}(x)\gamma_5\; e^{-\not{D}^2/M^2} (\not{D}+im_o)\psi(x) \rangle$$

$$= + \int d^4y \langle \delta^4(x-y)\ \mathrm{Tr}\ [\gamma_5\ \ e^{-\not{D}^2/M^2} (\not{D}+im_o)\psi(x)\ \bar{\psi}(y)]\rangle\ .$$

Using

$$(\not{D} + im_o) \langle \psi(x)\ \bar{\psi}(y) \rangle = -(\not{D} + im_o)\ G(x,y)$$
$$= i\ \delta^4(x-y),$$

this reduces to

$$i \int d^4y\ \ \delta^4(x-y)\ \ \mathrm{Tr}\left[\gamma_5\ \ e^{-\not{D}^2/M^2}\ \delta^4(x-y)\right]$$

$$= i \lim_{x\to y}\ \ \mathrm{Tr}\left[\gamma_5\ \ e^{-\not{D}^2/M^2}\ \delta^4(x-y)\right]. \qquad (6.110)$$

This is precisely what has been evaluated by Fujikawa. The second term on the right hand side of (6.109) also reduces to the expression (6.110).

Fujikawa [7] has given a derivation in which his operator whose eigenfunctions are chosen for expansion of ψ and $\bar{\psi}$ is $\not{D}$, but the regularization function f is arbitrary upto certain conditions. To see that this derivation is a special case of our work, we set $X = Y = \frac{\not{D}^2}{M^2}$ and $g(Y) = 1$ in Eq.(6.72). Then in Eq.(6.74), $N_1 = 1$, $N = 2N_1-1 = 1$. Thus we obtain a condition of the behavior of $f(x)$ and its derivatives from Eq. (6.74) as

$$b(q) < -\left(\frac{3+q}{2}\right) \qquad (6.111a)$$

In fact a closer examination of the derivation of Eq.(6.74) to the special case of our operator will show that the condition of Eq.(6.111a), while sufficient, is not necessary; but a weaker condition,

$$b(q) \; < - \left(\frac{2+q}{2} \right) \tag{6.111b}$$

is sufficient. This is so because the terms contributing have $\alpha+n \geq 2$ when the γ-matrix trace is required to not vanish. Eq.(6.111b) is still *not* the condition necessary in Fujikawa's derivation of anomalies. Explanation of this is as follows:

Fujikawa's regularization, being gauge-invariant must lead to an *overall* gauge-invariant $W^M_\nu[A]$. This, however, does not exempt $W^M_\nu[A]$ from having *individual* contributions which are gauge-variant and which we are insisting must be regularized, even though they may cancel out. In $\underline{\partial^\nu W^M_\nu[A]}$, it turns out that, such contributions simply do not arise. Hence a condition weaker than Eq. (6.111b) is sufficient, viz.,

$$b(q) \; < - \left(\frac{1+q}{2} \right) \tag{6.111c}$$

Translated in terms of $f^{(q)}(k^2)$, this reads that $f^{(q)}(k^2)$ falls of faster than k^{-q-1} as $|k| \longrightarrow \infty$. A closer examination of Fujikawa's work [7] would reveal that the conditions of Eq. (6.111c) are in fact necessary for his f also. [The conditions given by him viz $f(\infty) = f'(\infty) = f''(\infty) = \ldots\ldots = 0$ are necessary but not sufficient]. Further, as long as $X_1 = Y_1 = 0$, $W^1_\nu = 0$ and one obtains standard chiral anomaly; nomatter what f is (provided Eq. (6.111c) is satisfied). We have thus shown that Fujikawa's derivation of regularization independence of chiral anomaly is a special case of our work.

[2] Family of Anomalies derivation of chapter V

This is easily seen to be a special case of our derivation by setting

$$X = Y = \frac{\not{D}_a^2}{M^2}\,, \quad f(X) = g(X) = e^{-X}$$

with $\not{D}_a = \not{\partial} + iea\not{A}$. (6.112)

[3] Feynman Diagrammatic Method

The usual Feynman diagrammatic methods involve the triangle diagram, which needs to be regularized. This diagram can be regularized, in particular, by introducing a factor such as e^{-k^2/M^2} for a loop momentum k and then letting $M \longrightarrow \infty$. Such a regularization (though unconventional) can be looked upon as a special case of our results.

For example we may set in Eq. (6.3)

i) $X = \frac{\not{\partial}^2}{M^2} \quad f(X) = e^{-X} \quad g = 1\,,$

or ii) $Y = \frac{\not{\partial}^2}{M^2} \quad g(x) = e^{-x} \quad f = 1\,,$

or iii) $X = Y = \frac{\not{\partial}^2}{M^2} \quad f(x) = g(x) = e^{-x}$ etc. (6.113)

We may also alter the functions $f(x)$ and $g(x)$ as long as the diagram is regularized.

[4] Feynman Diagrammatic method with Pauli-Villars type regulator

In the triangle diagram, if we replace *one* of the propagators originating from the axial current vertex by replacing

$$\frac{1}{k^2 - m_o^2} \longrightarrow \frac{1}{k^2 - m_o^2} \quad \frac{M^2}{k^2 - M^2} \tag{6.114}$$

the Feynman diagram is regularized. This is also true if *both* the propagators originating from the axial-vector current vertex are similarly regularized. This Pauli-Villars type of regularization is obtain as in (6.113) with e^{-x} replaced by $\frac{1}{x+1}$.

[5] Proper time method

The expression for the anomaly in proper time method can be written as [39]

$$2A(x) = \lim_{s\to 0} \sum_n \varphi_n^\dagger (x) \quad e^{-i\not{D}^2 s}\gamma_5\, \varphi_n(x)$$

Analytic continuation in the variable s to complex value $s = -\frac{i}{M^2}$ puts the expression given above in the standard Fujikawa form [7] which is a special case of our results.

[6] Various other regularizations

In literature ψ and $\bar{\psi}$ have been expanded in different orthonormal bases [17,26]. We have always allowed X and Y to be generally different. The operators used in Refs. 17,26 can also be seen to be special cases of our treatment [In these works, however, attention is paid *only* to jacobians; in this respect our works [13,14,19,20] differ qualitatively from theirs, as has been

already emphasized before]. Non hermitian regularizations are also included as special cases of our treatment.

6.7 POINT-SPLITTING REGULARIZATION

We have dealt, so far, with the regularized axial-vector current defined by equation (6.3), with the operators of the type given in equation (6.4) and the regularizing functions as discussed in section (6.4). In this section we show that the point-splitting method can also be included in our treatment with the choice of operators (in equation (6.3)) $X = Y = \varepsilon.D_a$ and functions $f(X) = g(X) = e^{-X}$. These kind of operators are quite unlike the ones we have been discussing in the previous sections, so we examine the point-split current, in some detail, from the point of view of our treatment.

The current defined by the point-splitting technique is [6,34]:

$$W_\nu \equiv \lim_{\varepsilon\to 0} \left\langle \bar{\psi}\ (x + \tfrac{\varepsilon}{2})\gamma_\nu\gamma_5\ \psi(x - \tfrac{\varepsilon}{2})\ \exp\left\{ -iea\int_{x-\varepsilon/2}^{x+\varepsilon/2} A_\alpha(y)\,dy^\alpha\right\} \right\rangle$$

$$= \lim_{M\to\infty}\left[\mathrm{Tr}\left\{\int dz\,\delta^4(x-z)\,e^{\frac{\overleftarrow{\partial}.\hat{\eta}}{2M}}\, e^{\frac{-iea\hat{\eta}}{2M}.A+0[A]}\,\gamma_\nu\gamma_5\, e^{\frac{-iea\hat{\eta}}{2M}.A+0[A]}\, e^{\frac{-\hat{\eta}}{2M}.\partial}\, G(x-z)\right\}\right] \tag{6.115}$$

$$= \lim_{M\to\infty} W_\nu^M .$$

0 is an operator containing terms of $0(\varepsilon^3)$ and higher. $\varepsilon^\mu \equiv \frac{\hat{\eta}^\mu}{M}$, ($\hat{\eta}^2 = 1$), the limit $\varepsilon \to 0$ $(M \to \infty)$ understood as taken after

averaging over the directions of ε^{μ} $(\hat{\eta}^{\mu})$. 'a' is a real free parameter and a = 1 gives the gauge invariant definition of the point-split current.

It may be noted that the point-split current of equation (6.115) is of the form of eqn. (6.3) with

$$X = \frac{\hat{\eta}}{2M} \cdot D_a - \frac{1}{2}\left[\, iea\,\frac{\hat{\eta}.A}{2M},\ \frac{\hat{\eta}}{2M}\cdot\partial \,\right] + 0\left(\frac{1}{M^3}\right) + \ldots\ldots,$$

$$\overleftarrow{Y} = \overleftarrow{D}_a\cdot\frac{\hat{\eta}}{2M} - \frac{1}{2}\left[\,\overleftarrow{\partial}\cdot\frac{\hat{\eta}}{2M},\ -iea\,\frac{\hat{\eta}}{2M}\cdot A\,\right] + 0\left(\frac{1}{M^3}\right) + \ldots\ldots,$$

$$(6.116)$$

$$f(X) = e^{-X},$$

$$g(\overleftarrow{Y}) = e^{-\overleftarrow{Y}}.$$

This choice of X and Y differs from that of Eqs. (6.9) in two main aspects:

(1) There is a lack of manifest lorentz covariance with the choice given by Eq. (6.116). (This however is later taken care of by averaging over the directions $\hat{\eta}$ before taking the limit M —> ∞.) We will see below that the proof for the family structure of anomalies [defined by Eq. (6.6)] that we have given for our regularization holds, but for some technical modifications, for the point-split current.

(2) The series of terms in X was finite in Eq. (6.4) but here we have an infinite series: It will be easily seen towards the end of the section that it is sufficient to consider the finite, truncated series

$$X = \frac{\hat{\eta}}{2M} \cdot D_a \qquad \overleftarrow{Y} = \overleftarrow{D}_a \cdot \frac{\hat{\eta}}{2M} \tag{6.117}$$

i.e., we show that in the limit $M \longrightarrow \infty$ the choices given by eqns.(6.116) and (6.117) are equivalent.

We would thus be able to show that the point-splitting regularization is a special case of our treatment with the choice of aperator as in Eq. (6.117).

To show the family structure for the point-split current we follow the steps outlined in section (6.3). We split W_ν as

$$W_\nu = W_\nu^o + W_\nu^1 ,$$

$$\text{where} \quad W_\nu^o = \lim_{M\to\infty} \left(W_\nu^M \Big|_{a=1} \right)$$

Now, $\partial^\nu W_\nu$ has already been evaluated by detailed calculations elsewhere [34]; and we wish to show this result to be a special case of our treatment. As our emphasis in this section is to show that the derivation of the *family structure* in point-splitting method can be included in our treatment we shall assume the result for a = 1. We have

$$\partial^\nu W_\nu^o = 2im_o \, W_p^o + \frac{ie^2}{8\pi^2} \, F_{\mu\nu} \, \tilde{F}^{\mu\nu} \tag{6.118}$$

where $W_p = \lim_{M\to\infty} \langle \bar{\psi} e^{-\overleftarrow{Y}} \gamma_5 \, e^{-X} \psi \rangle$. [This equation should contain $W_p^o \equiv W_p \Big|_{a=1}$ but, W_p turns out to be independent of 'a' here also, using the same arguments as those used in section (6.3)]. Also since W_ν^o is gauge invariant,

$$\partial^{\mu} \frac{\delta}{\delta A_{\mu}} W^{o}_{\nu} = 0 \ . \tag{6.119}$$

Following theorems I and II of section (6.3) we will see that the family of anomalies follows if W^1_ν is local. (W^1_ν is lorentz invariant after "$\hat{\eta}$ averaging"). This is shown by examining the form of a typical contribution to W^1_ν.

Considering the general nth-order contribution of the Green's function (see eqn. (6.24)) in the current given by Eq. (6.115), a general contribution to W^M_ν can, upto factors, be written as

$$\mathrm{Tr}\left[\int d^4z d^4q_1 \ldots d^4q_n \, e^{iq_1(x-z)} e^{-i(q_2+\ldots+q_n)\cdot z} \int d^4k \, e^{i(k-\frac{q_1}{2})\cdot\frac{\hat{\eta}}{M}} .1.\exp\left(-\overleftarrow{D}_a\cdot\frac{\hat{\eta}}{2M}+\ldots\right)\right.$$

$$\times\ \gamma_\nu\gamma_5 \exp\left(-\frac{\hat{\eta}}{2M}\cdot D_a+\ldots\ldots\right).1.\frac{1}{\not{k}-(\not{q}_1+m_o)}\,\tilde{\not{A}}(q_2)\,\frac{1}{\not{k}-(\not{q}_1+\not{q}_2+m_o)}\ldots\tilde{\not{A}}(q_n)$$

$$\times\ \frac{1}{\not{k}-(\not{q}_1+\ldots+\not{q}_n+m_o)} \ . \tag{6.120}$$

Our assertion is that, in the above expression, all contribution to W^1_ν in the limit $M \longrightarrow \infty$ (which are finite in number as indicated by equation (6.122) below), are local. Hence W^1_ν itself would be local.

To do this consider the k-integral in (6.120) and scale $k \longrightarrow Mk$

$$M^{(4-n)}\int d^4k \, e^{i\left(k-\frac{q}{2M}\right)\cdot\hat{\eta}} .1.\left\{\exp\left(-\overleftarrow{D}_a\cdot\frac{\hat{\eta}}{2M}+\ldots\right)\right\}\gamma_\nu\gamma_5\left\{\exp\left(-\frac{\hat{\eta}}{2M}\cdot D_a+\ldots\right)\right\}.1$$

$$\times\ \frac{1}{\not{k}-\frac{(\not{q}_1+m_o)}{M}}\,\tilde{\not{A}}(q_2)\ldots\ldots\tilde{\not{A}}(q_n)\,\frac{1}{\not{k}-\frac{(\not{q}_1+\ldots\not{q}_n+m_o)}{M}} \ . \tag{6.121}$$

We imagine expanding (6.121) in powers of q_i and m_o upto a sufficiently many but finite number of terms, as was done in section (6.3). We must then ensure that the k-integrals for the terms in the expansion are well-defined in the UV and IR regions. The latter would ensure locality of contributions to W_ν^1 (Proof is along the lines of Theorem III of sec (6.3)), and hence of W_ν^1 itself.

In order to see the Infra-red convergence we look at Δ_M, the M power of a typical term in equation (6.121) after the expansion in powers of q_i and m_o is carried out.

$$\Delta_M \quad = \quad 4-n - R_M - \eta \;, \qquad (6.122)$$

$$n-1,\; R_M \;,\; \eta \quad = 0,\; 1,\; 2,\; \ldots\ldots$$

where R_M are the powers of M coming from the regulator exponentials $\exp\left(-\frac{\hat{\eta}}{2M}\,.D_a\;+\ldots\ldots\right)$; and η has the same origin as in section (6.3).

The overall power of k of *this* specific term which determines the convergence in the IR region is

$$\Delta_k \quad = \quad 4-n-\eta' \quad = \quad \Delta_M + R_M \;+\; (\eta-\eta')\;. \qquad (6.123)$$

Now, we note the following

(i) For terms that survive in the limit $M \longrightarrow \infty$, $\Delta_M \geq 0$.

(ii) $\eta-\eta' \;\geq\; 0$, as is easily seen by the inspection of Eq. (6.121).

(iii) $R_M > 0$ for 'a'-dependent terms, i.e., for contributions to W^1_ν .

As a result, we conclude, from Eq. (6.123), that for a term contributing to W^1_ν , $\Delta_k > 0$ and hence, the k - integrals are IR-convergent.

As far as the UV region is concerned we have verified that the integrals that are non-zero (even for finite M) are all rendered ultraviolet finite by this regularization.

Thus, we have shown that contributions to W^1_ν exist in UV limit and are indeed seen to be local by an argument in Sec.(6.3). Hence W^1_ν is local.

We now show that for regularization of the axial-vector current by point splitting techniques, the choices of regularizing aperators given by equations (6.116) and (6.117) are equivalent. Denoting the current regularized by the choice given in Eq. (6.117), as $\tilde{W}^M_\nu$, we see that

$$W^M_\nu - \tilde{W}^M_\nu \sim \mathrm{Tr}\Big[\int d^4 z \delta^4(x-z) e^{-\overleftarrow{D}_a \cdot \frac{\hat{\eta}}{2M}} \gamma_\nu \gamma_5 \Big\{ \Big[\overleftarrow{\partial} \cdot \frac{\hat{\eta}}{2M} , -iea \frac{\hat{\eta}}{2M} \cdot A \Big] + \ldots + \Big[A \cdot \frac{\hat{\eta}}{2M} , \partial \cdot \frac{\hat{\eta}}{2M} \Big] + \ldots \Big\} \times e^{\frac{\hat{\eta}}{2M} \cdot D_a} G(x-z) \Big] \qquad (6.124)$$

It is easy to see that in the limit $M \longrightarrow \infty$, none of the terms on RHS contribute. In the above equation, G_o never contributes due to there being insufficient number of γ matrices for the trace.

The next term in the expansion of G, G_1 is order $(1/M^2)$ or worse [see equation (6.121)]. Terms in curly brackets are also of order $(1/M^2)$ or worse. The contribution possibly surviving in the lt $M \rightarrow \infty$ could comes only from the $O\ (1/M^2)$ term in G_1. This contribution is vanishing since tr $\gamma_5 \gamma_\nu \not{k}\, \not{A}\, \not{k} = 0$. G_2 etc. are of order $(1/M^3)$ or worse and will give vanishing contributions in the limit $M \rightarrow \infty$. This shows that the choices given by equations (6.116) and (6.117) are really equivalent.

6.8 CONCLUSIONS

We have given in this chapter, a very general formulation of anomalies and their family structure in QED in path-integral formulation, which is based on the definition of axial-vector current regularized in a very general manner. We have shown how anomaly derivations in QED can be carried out with a very general definition of path-integral. We have also seen that particular choices of the operators X, Y and functions f and g in the definition of the regularized current lead to anomaly derivations which are equivalent to the other known derivations of anomalies like those based on Feynman diagrammatic calculations, point-splitting techniques, proper time methods, Fujikawa's derivations using "Energy operators" etc.

This work also has significance purely from a regularization and renormalization point of view. General *field-dependent* regulators of the kind first introduced by Fujikawa in his derivation of chiral anomaly in path-integral formulation, and which are known to have some novel features

[52,53], have been studied rigorously and in mathematical detail with the example of anomalies in QED. Using such field dependent regularizations within the path-integral framework (i.e., by using very general field dependent operators to define the path integral and associated ill-defined quantities), we have seen how the ambiguities in anomaly formulations (which manifest as arbitrariness in anomaly expressions) are related to the arbitrariness in the definition of the path-integral. In fact, we will see in the last chapter of the thesis that not only ambiguities in anomalies, but general renormalization prescription ambiguities can be accounted for by the *inherent* ambiguity in the definition of the path-integral. (This ambiguity in the definition is due to the arbitrariness in the choice of basis in which the fermionic fields are expanded). This will be done by discussing the renormalization of fermionic bilinear composite operators in the context of Yukawa type theory at one-loop level[27].

CHAPTER - VII

COVARIANT AND CONSISTENT ANOMALIES IN TWO DIMENSIONS IN PATH-INTEGRAL FORMULATION

7.1 INTRODUCTION

Our focus so far has been on the chiral and vector anomalies in QED. We have given a derivation of this family of anomalies in the path-integral formulation. In doing so we also saw the relation between the arbitrariness in anomaly expressions and the arbitrariness in the definition of the fermionic path-integral. Another example of ambiguities in anomaly, formulations is provided by the 'consistent' and 'covariant' anomalies associated with gauge currents in non-abelian gauge theories. Non-abelian anomalies have been studied extensively [21,24-26,41-49]. The anomaly (in the gauge source current) is usually defined as the gauge variation of the connected vacuum functional in the presence of external gauge fields, and its presence signals the breakdown of gauge invariance of the vacuum functional. The anomaly so defined [21], (the 'consistent' anomaly), is subject to certain consistency conditions, the Wess-Zumino consistency conditions [24] which put restrictions on its functional form. The associated current is the 'consistent' current. For the non-singlet, non-abelian chiral anomaly the consistency conditions restrict the anomaly from having a covariant expression. On the other hand, the non-singlet anomaly

may be obtained via a regularization that is gauge covariant. Then the anomaly is the 'covariant' anomaly having a covariant form, and the associated current is called the 'covariant' current. Bardeen and Zumino [21] have shown that these two currents are related to each other by addition of a local polynomial in gauge fields.

The covariant and consistent currents have been discussed a great deal [9,23,25,26,45-49] within Fujikawa's path-integral formulation. As has been said before, in this formulation anomalies are shown to arise from the jacobian for the Fermion measure under appropriate local transformations[1]. The works using Fujikawa technique generally attempt to find an appropriate regularization for such jacobian contributions that will reproduce the required forms for anomalies.

The previous works on covariant and consistent anomalies in non abelian chiral gauge theories are somewhat unsatisfactory on two counts. Firstly, these attempts seem to us quite artificial, in that available possible regularizations of jacobians are played with until one that reproduces the known result is arrived at. More importantly, they all assume Fujikawa identification, arrived at with an unregularized (therefore suspect) argument that anomaly is entirely from jacobian. These arguments have been made more rigorous but only for certain special covariant regularizations [48]. As has been emphasized

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[1]See however, the point emphasized in the next paragraph, below.

(iii) Operator chosen to regularize, $\not{D}_\alpha$ is hermitian. (iv) We take into account the contribution to anomaly other than that from jacobian when regularization using operators other than $\not{D}$ or its powers is used unlike previous works. Consequently (v) *Same* operator is used in determining the bases to expand *both* ψ_L and $\bar{\psi}_L$, and also to regularize expressions for currents, unlike Ref. 25, 26. (vi) While one can always obtain a consistent current by adding local terms to covariant current, such consistent current is not classically conserved [21,26]. Our currents for all 'α' are classically conserved.

In section (7.2) we define our theory, and give the consistency condition. In section (7.3), we write down the most general local family of anomaly equation satisfied by a chiral current in our theory, and give a regularization for the current that reproduces this family. In section (7.4) we prove our results for this family structure. We note that special choices of the free parameter yield covariant and consistent currents. In section (7.5) we give another, and indirect proof that the consistent current we have formed is indeed so. We rely on our previous works [13, 20] for the abelian case to obtain corresponding results here. In section (7.6) we make some concluding remarks related to our results, and some of the other works [25,26].

7.2 PRELIMINARY

We work in 2-dimensional Euclidean chiral gauge theory given by the action

$$S = \int d^2x\, \bar{\psi}_L\, (i\partial\!\!\!/ - A\!\!\!/)\, \gamma_L\, \psi_L \;\equiv\; \int d^2x\, \bar{\psi}_L i\, D\!\!\!\!/\, \gamma_L\, \psi_L . \tag{7.1}$$

Our convention for Dirac matrices is as given below.

γ_μ (μ = 1,4) are antihermitian Dirac matrices. The metric $g_{\mu\nu}$ = (-1,-1). The γ-matrices satisfy

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} + \varepsilon^{\mu\nu}\, \gamma_5 , \tag{7.2}$$

with $\varepsilon^{\mu\nu} = - \varepsilon^{\nu\mu}$ and $\varepsilon^{14} = -i$. $\gamma_5 = i\gamma^1\gamma^4$ is hermitian with $\gamma_5^2 = 1$. $\gamma_L \equiv \frac{1}{2}\,(1 - \gamma_5)$. ψ_L are left-handed chiral spinors.

And

$$A_\mu = \frac{\lambda^a}{2}\, A_\mu^a \tag{7.3}$$

are hermitian gauge field matrices, $\frac{\lambda^a}{2}$ being a representation of the simple gauge group G satisfying Lie-algebra

$$\left[\frac{\lambda^a}{2} , \frac{\lambda^b}{2} \right] = i f^{abc}\, \frac{\lambda^c}{2} , \tag{7.4}$$

with $f_{abc}\, f_{dbc} = \delta_{ad}$

The above theory has a classically conserved current

$$J_\mu^{a\ cl} = \bar{\psi}_L\, \frac{\lambda^a}{2}\, \gamma_\mu\, \gamma_L\, \psi_L . \tag{7.5}$$

The consistent anomaly satisfies an algebraic condition, viz., the Wess-Zumino consistency condition [24]. Let the anomaly be

defined by

$$D^{\mu} \; J^{a}_{\mu} \;=\; G^{a} \;. \tag{7.6}$$

We define

$$\Lambda \;=\; \Lambda^{a} \frac{\lambda^{a}}{2} \;,$$

where Λ^{a} are infinitesimal parameters of gauge transformation. We define

$$T_{\Lambda} \;\equiv\; \int \Lambda^{a}(x) \; D^{ab}_{\mu} \; \frac{\delta}{\delta \; A^{b}_{\mu}(x)} \; d^{2}x \;. \tag{7.7}$$

Then the consistency condition reads [21]

$$\int d^{2}x \; (\; \Lambda'^{a} \; T_{\Lambda} \; G^{a} - \Lambda^{a} \; T_{\Lambda'} \; G^{a}) \;=\; \int d^{2}x \; [\Lambda, \; \Lambda']^{a} \; G^{a} \;. \tag{7.8}$$

The derivation of this condition in Ref. 21 also makes it clear that if J^{a}_{μ} is derivable from a generating functional $W[A]$, i.e.,

$$J^{a}_{\mu}(x) \;=\; - \; \frac{\delta W[A]}{\delta A^{a}_{\mu}(x)} \;, \tag{7.9}$$

then the above condition is automatically satisfied.

The consistent anomaly in the context of the present model is known to be of the form [21]

$$D^{\mu} \; J^{a}_{\mu} \;=\; c \; \varepsilon^{\mu\nu} \; \partial_{\mu} \; A^{a}_{\nu} \;. \tag{7.10}$$

7.3 GENERAL ANOMALY EQUATION AND REGULARIZED CURRENTS

In this section, we shall give the most general local anoma equation possible for a chiral current and then we shall gi definition of regularized currents that will reproduce this equati

(as shown in the next section). These definitions are of the same ki that have been used in our earlier works [14,19,20,22].

A. Most general anomaly equation

Let $J^a_\mu(x)$ be the covariant current. It satisfies the anomaly equation

$$D^\mu J^a_\mu(x) = c\, F^a_{\mu\nu}\, \varepsilon^{\mu\nu} , \tag{7.11}$$

where for our model $c = -\frac{1}{4\pi}$. The right-hand side of Eq. (7.11) is the covariant anomaly.

One can define a series of local currents $\underset{\sim}{J}{}^a_\mu(x)$ which differ from the covariant current $J^a_\mu(x)$ by finite local terms in gauge fields. The local modifications of J^a_μ that preserve the global gauge invariance and Lorentz invariance are only of two kinds taking into account the dimension of J^a_μ , viz.,

$$\underset{\sim}{J}{}^a_\mu(x) \equiv J^a_\mu(x) + c \left[\xi\, A^a_\mu(x) + \eta\, \varepsilon^\nu_\mu\, A^a_\nu(x) \right]. \tag{7.12}$$

The second term can be put as J^a_μ has no definite parity property.

As a result, the most general local anomaly equation is

$$D^\mu \underset{\sim}{J}{}^a_\mu(x) = c \left\{ F^a_{\mu\nu}\, \varepsilon^{\mu\nu} + \xi\, \partial.A^a + \eta\, \varepsilon^{\mu\nu} \left[\partial_\mu A^a_\nu - f^{abc} A^b_\mu A^c_\nu \right]\right\} , \tag{7.13}$$

and, thus, is a two parameter equation. The consistent anomaly equation for the consistent current $\tilde{J}^a_\mu(x)$ is generally defined by [21]

$$D^{\mu} \quad \tilde{J}^{a}_{\mu}(x) \quad = \quad c \; \varepsilon^{\mu\nu} \; \partial_{\mu} \; A^{a}_{\nu} \; , \tag{7.14}$$

but the consistency condition of Eq. (7.8) leaves the freedom to define a consistent current as

$$\approx{J}^{a}_{\mu}(x) \quad = \quad \tilde{J}^{a}_{\mu}(x) \quad + \quad c \; \xi \; A^{a}_{\mu}(x) , \tag{7.15}$$

as the last term in Eq. (7.15) can be derived from a local addition, $W_1[A]$, to the generating functional $W[A]$, from which consistent current $\tilde{J}_{\mu}{}^{a}(x)$ is defined via Eq.(7.9) where

$$W_1[A] \quad = \quad \frac{1}{2} \; \xi c \int d^2x \quad A^{a}_{\mu} \quad A^{a\mu} , \tag{7.16}$$

so that

$$\overset{\approx}{J}{}^{a}_{\mu}(x) \quad = \quad \frac{\delta}{\delta A^{a}_{\mu}(x)} \left[\; W[A] \; + \; W_1[A] \; \right] \tag{7.17}$$

is also a consistent current, as seen in section (7.2). [Note, however, that the last term in Eq. (7.12) cannot be derived in a similar manner and hence cannot be treated as an ambiguity in $\tilde{J}^{a}_{\mu}(x)$]. This consistent current satisfies the anomaly equation

$$D^{\mu} \; \overset{\approx}{J}{}^{a}_{\mu}(x) \quad = \quad c\left[\varepsilon^{\mu\nu} \; \partial_{\mu} \; A^{a}_{\nu} \; + \; \xi \; \partial.A^{a}\right] \; . \tag{7.18}$$

In this work, we are interested in a chiral current. We define a current in two dimension to be left chiral if

$$(\; g^{\mu\nu} \; - \; \varepsilon^{\mu\nu} \;) \; J^{a}_{\nu} \quad = \quad 0 \; . \tag{7.19}$$

Our unregularized current $\bar{\psi}_L \frac{\lambda a}{2}\gamma_{\mu} \; \gamma_L \; \psi_L$ satisfies this

condition. Now this condition restricts the freedom in Eq. (7.12) because

$$\left(q_{\alpha}{}^{\mu} - \varepsilon_{\alpha}{}^{\mu} \right) \left[\underset{\sim}{J}{}^{a}_{\mu}(x) - J^{a}_{\mu}(x) \right] = 0 = c\left[\xi (A^{a}_{\alpha} - \varepsilon_{\alpha}{}^{\mu} A^{a}_{\mu}) + \eta (\varepsilon_{\alpha}{}^{\mu} A^{a}_{\mu} - A^{a}_{\alpha}) \right],$$

requiring $\eta = \xi$. As a result, for a left chiral current, the most general local anomaly equation is, from Eq. (7.13),

$$D^{\mu} \underset{\sim}{J}{}^{a}_{\mu}(x) = c\left[\varepsilon^{\mu\nu} F^{a}_{\mu\nu} + \xi \left\{ \partial . A^{a} + \varepsilon^{\mu\nu} \partial_{\mu} A^{a}_{\nu} - \varepsilon^{\mu\nu} f^{abc} A^{b}_{\mu} A^{c}_{\nu} \right\} \right]. \tag{7.20}$$

We note that $\underset{\sim}{J}{}^{a}_{\mu}$ given above satisfies the consistent current divergence equation (7.18) for $\xi = -1$.

B. General definition of regularized current

Our aim in this work is to give definition of regularized chiral current such that its divergence produces the one-parameter family of Eq. (7.20), and in particular obtain the consistent and covariant currents as special cases.

To this end, we shall find it sufficient to consider the one parameter family of operators $\not{D}_{\alpha} = \not{\partial} + i\alpha \not{A}$, as was done in Ref. [14], and analogously define

$$J^{aM}_{\mu}(x) = \bar{\psi}_{L}(x) \exp\left[-\frac{\not{D}^{2}_{\alpha}}{M^{2}} \right] \frac{\lambda^{a}}{2} \gamma_{\mu}\gamma_{L} \exp\left[-\frac{\not{D}^{2}_{\alpha}}{M^{2}} \right] \psi_{L}(x). \tag{7.21}$$

Evidently for $\alpha = 1$, we obtain the covariant current. For

$$J_\mu^{aM}(x) = \bar{\psi}_L(x) \exp\left[-\frac{\overleftarrow{\not{D}}^2}{M^2}\right] \frac{\lambda^a}{2} \gamma_\mu \gamma_L \exp\left[-\frac{\not{D}^2}{M^2}\right] \psi_L(x) \qquad (7.22)$$

is a covariant vector under vector gauge transformations. We shall show in the next section that for an arbitrary α we reproduce the family equation (7.20) with

$$\xi = -(1-\alpha),$$

and for $\alpha = 0$ we obtain the consistent current satisfying Eq. (7.18).

7.4 FAMILY EQUATION AND CONSISTENT CURRENT

In this section we shall derive the family equation (7.20) and in particular the consistent anomaly equation (7.18) (for $\xi = -1$). We shall follow the method given in chapter V [20]. We define

$$W_\nu^{aM}[A] \equiv \langle J_\nu^{aM} \rangle$$

$$W_\nu^{a}[A] = \lim_{M\to\infty} W_\nu^{aM}[A] \qquad (7.23)$$

$$W_\nu^{ao}[A] = W_\nu^{a}[A]\Big|_{\alpha=1} \qquad (7.24)$$

$$W_\nu^{a}[A] \equiv W_\nu^{ao}[A] + W_\nu^{a1}[A] \qquad (7.25)$$

The proof of the family equation (7.20) proceeds by proving the following results whose proofs are indicated later.

Theorem I : $W_\nu^{ao}[A]$ satisfies the covariant current divergence equation $D^\nu W_\nu^{ao} = c\, \varepsilon^{\mu\nu} F_{\mu\nu}^a$. (7.26)

Theorem II : $W_\nu^{a1}[A]$ is a local polynomial in A_μ^a, of dimension one.

Theorem III: $(g_\mu{}^\nu - \varepsilon_\mu{}^\nu)\, W_\nu{}^{a1}[A] \;=\; 0.$ (7.27)

From theorems II and III it follows that

$$W_\nu^{a1}[A] \;\equiv\; c\xi\, [A_\nu^a + \varepsilon_\nu{}^\alpha\, A_\alpha^a\,]. \tag{7.28}$$

Theorem IV : $c \;=\; -\,\frac{1}{4\pi}$.

$$\xi \;=\; -\,(1-\alpha). \tag{7.29}$$

Theorem V : $W_\nu^a[A]$ satisfies the family equation (7.20):

$$D^\nu W_\nu^a[A] \;=\; c\left[\, \varepsilon^{\mu\nu}F_{\mu\nu}^a \;-\; (1-\alpha)\,\left\{\partial.A^a + \varepsilon^{\mu\nu}\partial_\mu A_\nu^a \;-\; \varepsilon^{\mu\nu} f^{abc}\, A_\mu^b A_\nu^c\,\right\}\right]. \tag{7.30}$$

The proofs of these theorems are as follows.

Proof of Theorem I : We observe that

$$\partial^\nu W_\nu^{ao}[A] = -\langle\bar\psi_L e^{-\overleftarrow{\not{D}}^2/M^2}(\tfrac{\lambda^a}{2})\,(\overleftarrow{\not{D}}\gamma_L)\, e^{-\not{D}^2/M^2}\psi_L\rangle + \langle\bar\psi_L e^{-\overleftarrow{\not{D}}^2/M^2}\,(\tfrac{\lambda^a}{2})\,\not{D}\gamma_L e^{-\not{D}^2/M^2}\,\psi_L\rangle \tag{7.31}$$

Using $\langle\, \bar\psi_L(z)\psi_L(x)\,\rangle = -\,\gamma_L\, \langle\psi(x)\bar\psi(z)\,\rangle\gamma_R \equiv \gamma_L G(x,z)\gamma_R$

$$= G(x,z)\gamma_R \;=\; \gamma_L G(x,z), \tag{7.32}$$

(where $G(x,z)$ is defined by $\not{D}G(x,z) = -i\delta(x-z)$)

in Eq. (7.31), and observing that $\frac{\lambda a}{2}\not{\mathfrak{D}} = \not{\mathfrak{D}}\frac{\lambda a}{2} - f^{abc}(\frac{\lambda c}{2})\not{A}^{b}$,

we have

$$\partial^{\nu}W_{\nu}^{ao} - f^{abc}A^{\nu b}W_{\nu}^{co} = -\mathrm{Tr}\left[\int d^{2}z\ G(x-z)\ \not{\mathfrak{D}}\ e^{-\not{\mathfrak{D}}^{2}/M^{2}}(\frac{\lambda a}{2})\gamma_{L}\ e^{-\not{D}^{2}/M^{2}}\delta^{2}(x-z)\right]$$
$$+\mathrm{Tr}\left[\int d^{2}z\ \delta^{2}(x-z)\ e^{-\not{\mathfrak{D}}^{2}/M^{2}}(\frac{\lambda a}{2})\gamma_{R}\ e^{-\not{D}^{2}/M^{2}}\not{D}\ G(x-z)\right]$$

(7.33)

which gives

$$\partial^{\nu}W_{\nu}^{ao} - f^{abc}A^{\nu b}W_{\nu}^{co} = -i\mathrm{Tr}\int d^{2}z\ \delta^{2}(x-z)\ e^{-\not{\mathfrak{D}}^{2}/M^{2}}(\frac{\lambda a}{2})(-\gamma_{L}+\gamma_{R})\ e^{-\not{D}^{2}/M^{2}}\delta^{2}(x-z)$$

(7.34)

The RHS of Eq. (7.34) is exactly of the form of regularized "chiral jacobian" evaluated by Fujikawa [7] and it is easily seen that such a term will be of the form $\varepsilon^{\mu\nu}F_{\mu\nu}^{a}$.

Proof of Theorem II: Proceeds exactly as in chapter V [20], done there for the abelian case.

Proof of Theorem III: Our regularization preserves the chiral condition (7.19). Hence

$$(g_{\mu}^{\nu} - \varepsilon_{\mu}^{\ \nu})\ W_{\nu}^{aM} = 0 \qquad (7.35)$$

Take limit $M \longrightarrow \infty$. Eq. (7.27) then holds.

Proof of Theorem IV : We compare this case with the corresponding abelian case dealt with in chapters II and III [13,14]. For

massless case the results of chapters II and III (see Eqs. (2.51), (2.52) or Eqs. (3.18)) read

$$\partial^{\mu} \langle J^{V}_{\mu} \rangle = \frac{(1-\alpha)}{2\pi} \partial^{\mu} A_{\mu}$$

$$\partial^{\mu} \langle J^{A}_{\mu} \rangle = \frac{(1+\alpha)}{2\pi} \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu} \tag{7.36}$$

Hence $J_{\mu} = \frac{1}{2} (J^{V}_{\mu} - J^{A}_{\mu})$ will satisfy

$$\partial^{\mu} \langle J_{\mu} \rangle = - \frac{1}{2\pi} \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu} + \frac{(1-\alpha)}{4\pi} (\partial . A + \varepsilon^{\mu\nu} \partial_{\mu} A_{\nu}) \tag{7.37}$$

Comparing this equation with the equation (7.26) for $\alpha=1$ and specific group index, the corresponding non-abelian analogue becomes

$$\partial^{\nu} W^{ao}_{\nu}[A] = 2c\, \varepsilon^{\mu\nu} \partial_{\mu} A^{a}_{\nu} ,$$

where we have noted the replacement $1 \longrightarrow f_{abc}\, f_{dbc} \equiv \delta_{ad}$,for the non abelian case in the only contributing term. Thus, we obtain

$$c = - \frac{1}{4\pi} .$$

We further obtain,

$$\partial^{\nu} W^{a}_{\nu} = \partial^{\nu} [W^{ao}_{\nu} + W^{a1}_{\nu}]$$

$$= \frac{-1}{4\pi} \left[2\varepsilon^{\mu\nu} \partial_{\mu} A^{a}_{\nu} + \xi \left\{ \partial . A^{a} + \varepsilon^{\mu\nu} \partial_{\mu} A^{a}_{\nu} \right\} \right] . \tag{7.38}$$

Comparing this with Eq. (7.37) we obtain

$$\xi \;=\; -(1-\alpha)\ . \tag{7.39}$$

Substituting these values of c and ξ, the proof of theorem V follows in the same way as on obtains Eq. (7.13) from Eq. (7.12) and one gets

$$D^{\nu}W^{a}_{\nu}[A] \;=\; \frac{-1}{4\pi}\left\{\varepsilon^{\mu\nu}F^{a}_{\mu\nu} \;-\; (1-\alpha)\;\left[\partial.A^{a}+\varepsilon^{\mu\nu}\partial_{\mu}A^{a}_{\nu} \;-\; \varepsilon^{\mu\nu}f^{abc}A^{b}_{\mu}\;A^{c}_{\nu}\right]\right\}\ . \tag{7.40}$$

In particular for $\alpha = 0$, one obtains the left chiral consistent current divergence equation

$$D^{\nu}W^{a}_{\nu}[A]_{con} \;=\; -\;\frac{1}{4\pi}\;\left\{\;\varepsilon^{\mu\nu}\partial_{\mu}A^{a}_{\nu} \;-\; \partial.A^{a}\;\right\}\ . \tag{7.41}$$

7.5 DIRECT PROOF FOR THE EXPRESSION OF CONSISTENT CURRENT

In this section, we shall give a direct proof that the regularized current of Eq. (7.21) with $\alpha = 0$ viz,

$$\approx\!\!J^{aM}_{\mu} \;=\; \bar{\psi}_{L}(x)\;\exp\left[-\;\frac{\overleftarrow{\not{D}}^{2}}{M^{2}}\right]\;\frac{\lambda^{a}}{2}\;\gamma_{\mu}\gamma_{L}\;\exp\left[-\;\frac{\not{D}^{2}}{M^{2}}\right]\;\psi_{L}(x)\,, \tag{7.42}$$

is a consistent current. As seen in Sec. (7.2), any current J^{a}_{μ} derivable from a generating functional W[A] via

$$\langle\; \approx\!\!J^{a}_{\mu}\;\rangle \;=\; -\;\frac{\delta\;\tilde{W}[A]}{\delta\;A^{a}_{\mu}(x)}\ ,$$

satisfies the consisting condition of Eq. (7.8).

To do this we consider

$$\tilde{Z}^{M}[A]=\int D\psi_{L}D\bar{\psi}_{L}\;\exp\left\{\int\left[\bar{\psi}_{L}\;i\not{\partial}\gamma_{L}\psi_{L}-\bar{\psi}_{L}\;\exp\left[-\;\frac{\overleftarrow{\not{D}}^{2}}{M^{2}}\right]\not{A}\gamma_{L}\;\exp\;\left[-\;\frac{\not{D}^{2}}{M^{2}}\right]\;\psi_{L}\right]\;d^{4}x\right\}$$

$$\equiv\;\int D\psi_{L}D\bar{\psi}_{L}\;\exp\;S^{R} \tag{7.43}$$

$$- \frac{\delta \tilde{Z}^M[A]}{\delta A^a_\mu(x)} = \int D\psi_L D\bar{\psi}_L \, \approx{J}^{aM}_\mu(x) \exp S^R$$

Writing $\tilde{W}^M[A] = \ln \tilde{Z}^M[A]$, we have

$$\frac{\delta \tilde{W}^M[A]}{\delta A^a_\mu(x)} = - \frac{\int D\psi_L \bar{D}\psi_L \, \approx{J}^{aM}_\mu(x) \, e^{S^R}}{\int D\psi D\bar{\psi} \quad e^{S^R}} . \tag{7.44}$$

Comparing the expression of Eq. (7.44) with $W^{aM}_\nu[A]_{con}$ of Eq. (7.23) obtained by setting $\alpha = 0$,

$$W^{aM}_\nu[A]_{con} = \frac{\int D\psi_L \bar{D}\psi_L \, \approx{J}^{aM}_\mu (x) \, e^S}{\int D\psi_L D\bar{\psi}_L \quad e^S} . \tag{7.45}$$

which differs from that of the right-hand side of Eq. (7.44) only in that the regularized action S^R is replaced by unregularized action S. However, as shown below, this does not affect the expression for the anomaly.

Firstly, we note that the only term in the expansion of right-hand side of Eq. (7.45) in powers of A, for which regularization matters at all, is that involving its first derivative with respect to A, viz $\frac{\delta W^a_\nu}{\delta A^b_\mu(y)}$. It is easily seen that in the expression that contributes to this term (which is just a regularized Feynman diagram, because our regulator is field independent here) each Green's function in momentum space becomes replaced by

$$\frac{1}{k} \; e^{-k^2/2M^2} \qquad \text{from} \quad \frac{1}{k} \; e^{-k^2/M^2}$$

as one goes from expression (7.44) to expression (7.45). But as this term is finite, its limit as $M \longrightarrow \infty$ is independent of this change $M \longrightarrow M/\sqrt{2}$. Hence

$$W_\nu^a\,[A]_{con} = -\frac{\delta\,\tilde{W}\,[A]}{\delta\,A_\mu^a(x)}\,, \tag{7.46}$$

proving that the case $\alpha = 0$, does yield a consistent current.

7.6 CONCLUDING REMARKS

We notice that the covariant divergence of the general chiral current defined in Eq. (7.21) can be written as

$$\begin{aligned} D^\nu W_\nu^{aM} \equiv (D^\nu \langle J_\nu^M \rangle)^a = &- \langle \bar{\psi}_L\, e^{-\bar{\not{D}}_\alpha^2/M^2}\, \bar{\not{D}}_\alpha\, \frac{\lambda^a}{2}\, \gamma_L\, e^{-\not{D}_\alpha^2/M^2}\, \psi_L \rangle \\ &+ \langle \bar{\psi}_L\, e^{-\bar{\not{D}}_\alpha^2/M^2}\, \frac{\lambda^a}{2}\, \gamma_R\, \not{D}_\alpha\, e^{-\not{D}_\alpha^2/M^2}\, \psi_L \rangle \\ &- (1-\alpha) f^{abc} A^{\nu b} W_\nu^{cM} \;. \end{aligned} \tag{7.47}$$

Noticing that $\not{D}_\alpha = \not{D} - i(1-\alpha)\not{A}$ and that $\not{D}_\alpha\, e^{\not{D}_\alpha^2/M^2} = e^{\not{D}_\alpha^2/M^2}\, \not{D}_\alpha$, and using Eq. (7.32),

Eq. (7.47) can be seen to be of the form

$$D^\nu W_\nu^{aM} \sim \mathrm{Tr}\int dz\; \delta^2(x-z)\; e^{-\bar{\not{D}}_\alpha^2/M^2}\, \frac{\lambda^a}{2}\, \gamma_5\, e^{-\not{D}_\alpha{}^2/M^2}\, \delta^2(x-z) + (1-\alpha)\ \text{terms} \tag{7.48}$$

The first term on the right-hand side of Eq. (7.48) is a "chiral

jacobian" term already dealt with in section (7.4) for $\alpha = 1$ (see Eq. (7.34)). This term evaluated in the limit $M \to \infty$ would give an answer obtained by replacing A by αA in $c\varepsilon^{\mu\nu}F^a_{\mu\nu}$. We notice two things from the equations above.

1. The $\alpha = 1$ case, which gives the covariant current divergence in Eq. (7.48) has the anomaly coming entirely from jacobian term.
2. When $\alpha = 0$, Eq. (7.48) gives the consistent current divergence and the anomaly contribution comes entirely from the $(1-\alpha)|_{\alpha=0}$ terms. Such terms can be interpreted, in the path-integral formulation as coming from the action, if we follow the algebra of chapter II [13] for computing the current divergence. The jacobian term being necessarily proportional to α, gives no contribution in this case.

This discussion highlights the statements we made in section (7.1), viz., that the anomaly should not be associated, *a priori*, with jacobian factors. Such an identification, made by formal derivations, is suspect. When this is done, the results for the consistent and covariant anomaly are obtained in a very unsatisfactory way, in that all kinds of regularizations and regularizing operators (for regularizing the jacobians) are played around with till the desired results are obtained. In Ref. 26 for example the consistent anomaly is derived by regularizing *separately* the jacobians associated with measure $D\psi_L$ and $D\bar{\psi}_L$ in terms of different and rather unusual non-covariant combinations (as the authors themselves concede) of non hermitian 'Weyl operators'. Ref. 25 employs a covariant regularization using

combinations of similar Weyl operators to derive the covariant form of the anomaly. In contrast, the path-integral prescription of ours is very straightforward. As shown below, we expand ψ_L and $\bar{\psi}_L$ in bases of *same* operator $\not{D}_\alpha^2$. Definition of the integration measure *and* the regularization is in terms of the hermitian $\not{D}_\alpha$. The covariant and consistent currents are obtained as special cases of a *single* regularized current.

Explicitly, the path-integral prescription which has so far only been implied, is as follows.

The fermionic fields are expanded in terms of the basis of hermitian operator $\not{D}_\alpha$. The eigenvalue equation for $\not{D}_\alpha$ is

$$\not{D}_\alpha \phi_n = \lambda_n \phi_n . \tag{7.49}$$

Define a chiral orthonormal basis [8]

$$\begin{aligned}
\phi_n{}^L &= \frac{1-\gamma_5}{\sqrt{2}} \phi_n \qquad && \lambda_n > 0 \\
&= \frac{1-\gamma_5}{2} \phi_n && \lambda_n = 0 \\
\phi_n{}^R &= \frac{1+\gamma_5}{\sqrt{2}} \phi_n && \lambda_n > 0 \\
&= \frac{1+\gamma_5}{2} \phi_n && \lambda_n = 0
\end{aligned} \tag{7.50}$$

$$\begin{aligned}
\not{D}_\alpha{}^2 \phi_n{}^L &= \lambda_n{}^2 \phi_n{}^L . \\
\not{D}_\alpha{}^2 \phi_n{}^R &= \lambda_n{}^2 \phi_n{}^R .
\end{aligned} \tag{7.51}$$

The fields ψ_L and $\bar{\psi}_L$ are expanded as

$$\psi_L = \sum_{\lambda_n \geq 0} a_n \phi_n{}^L(x) \tag{7.52a}$$

$$\bar{\psi}_L = \sum_{\lambda_n \geq 0} \bar{b}_n \phi_n{}^{R^\dagger}(x) \tag{7.52b}$$

The path-integral measure is *defined* as

$$\prod_m d\bar{b}_m \prod_n da_n$$

The following orthonormality relations are assumed

$$\int d^2x\, \phi_n{}^{L\dagger} \phi_m{}^L = \delta_{n,m} = \int \phi_n{}^{R\dagger} \phi_m{}^R\, d^2x\ . \tag{7.53}$$

The chiral current of Eq. (7.21) is

$$J_\nu^{aM}(x) = \sum_{\substack{p,q \\ \lambda_p, \lambda_q \geq 0}} \bar{b}_p\, a_q\, X_{pq\nu}^{\ a}(x)\ , \tag{7.54}$$

where

$$X_{pq\nu}^{\ a}(x) = \phi_p{}^{R\dagger} \frac{\lambda^a}{2} \gamma_\nu \gamma_L \phi_q{}^L(x)\, e^{-\lambda p^2/M^2}\, e^{-\lambda q^2/M^2}\ . \tag{7.55}$$

The Green's function is carefully defined by summing over the non-zero eigen modes of $i\not{D}_\alpha$ [50].

CHAPTER VIII

AN INTERPRETATION OF RENORMALIZATION PRESCRIPTION AMBIGUITIES IN PATH-INTEGRAL FORMULATION

8.1 INTRODUCTION

A common feature of all Quantum Field Theories (QFTs) which have been phenomenologically successful is that they are renormalizable. Renormalizability is a very desirable feature to have in a QFT, for it gives the theory immense predictive power. QFT are unavoidably plagued with ultraviolet divergences in the calculation of physically observable quantities [18,38,51] and therefore to yield physically meaningful results, it is desirable that they be renormalizable. In a renormalizable QFT, the generic divergences reside in a finite number of primitive proper vertices. The renormalization procedure defines by hand these vertices to have given finite values, and shows then that all physical observables are thereby rendered finite. Renormalization procedure can also be understood as subtraction of infinities. When an infinity is subtracted from an infinity the remaining finite part becomes, ambiguous. Thus the finite parts of proper vertices that have divergences are ambiguous. These ambiguities are normally looked upon as a consequence of the arbitrariness in regularization and subtraction procedure and are not associated with a "meaning". These ambiguities, of course, cannot have a "physical meaning" because they have no observable consequences in

renormalizable theories. The aim of the work in this chapter is to give a possible significance to these renormalization ambiguities in the path-integral formulation [27].

We have already, this far in the thesis, had an understanding of ambiguities in anomaly formulations within the path-integral formulation. The ambiguities in anomalies is a result of the (potential) divergences or ill-defined nature of certain quantities (triangle-diagram, anomalous jacobian etc) which calls for regularization and renormalization. The ambiguities manifest as arbitrariness in the expressions for anomalies. There are many examples of this. We have already discussed some of these. For example, we discussed the family structure of anomalies in $(QED)_{2,4}$. In non-abelian gauge theories, anomalies in currents can be covariant, consistent (or neither) [9,21,23]. Working in a path- integral framework inspired by that of Fujikawa [7] we have explained these ambiguities by using operators other than Fujikawa's "Energy Operators" to define the path-integral [13,14,19,20,22,23]. We have shown that these ambiguities can be seen to arise from the arbitrariness in the definition of the path-integral. From a general view-point, the path-integral measure could be defined in terms of *any* complete set of eigenfunctions anyhow.

The aim of the present work is to explore whether all the renormalization ambiguities (and not just the ones associated with anomalies) can be understood as arising entirely from the arbitrariness in the definition of the path-integral. We do this by considering the renormalization of bilinear composite operators

in path-integral framework, in the context of Yukawa theory. We choose this theory to avoid questions of gauge invariance (which are irrelevant to the issue anyway). Our treatment is limited to one loop (as are practically all works using Fujikawa's path-integral formulation [52]).

In section (8.2) we give definition of a regularized and renormalized fermionic bilinear composite operator, $\langle O^R \rangle$, in our path-integral framework. In section (8.3), with the help of two simple operators, a scalar operator and a vector operator, we illustrate our point, that all ambiguities in $\langle O^R \rangle$ can be correlated with the arbitrariness in the choice of basis in the definition of the path-integral and can be interpreted as arising from it. In section (8.4) the results of the previous section are made more rigorous by considering renormalization of an arbitrary fermionic bilinear operator. In section (8.5) we give conclusions.

8.2 PRELIMINARIES

We shall work in the context of Yukawa theory with Euclidean action[1]

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[1] Our focus in this article is on renormalization; so we directly consider Euclidean action given below. This action transformed to Minkowski space may lack Hermiticity, CP and T invariance, but we have ignored it because it is irrelevant to the issue. These can be corrected at the cost of slightly complicated calculation.

$$S \equiv \int d^4x \, \mathcal{L} = \int d^4x \, [\bar{\psi} \, i\not{\partial} \, \psi + ig\bar{\psi} \, \psi \, \phi - m_o \bar{\psi} \, \psi \,] \tag{8.1}$$
$$\equiv \int d^4x \, \bar{\psi} \, (i\not{D} - m_o) \, \psi \, .$$

Here ϕ is treated as an external, real, scalar field. The generating function of one fermion loop diagrams with external scalar lines is defined by

$$W[\phi] = \int D\psi \, D\bar{\psi} \, e^S. \tag{8.2}$$

To define the fermion measure we must choose an operator $X[\phi]$ which has a complete set of eigenfunctions.We could choose $X[\phi]$ to be a Hermitian operator having a complete set of eigenfunctions. Or we could [22] write

$$X[\phi] = X_o' + X_1'[\phi] \, , \tag{8.3}$$

where X_o' is an hermitian operator independent of ϕ having a complete set of eigenfunctions, and $X_1'[\phi]$ vanishes at ϕ=0. One can then treat $X_1'[\phi]$ as a perturbation on X_o', and can construct a pertubative complete set of eigenfunctions of $X[\phi]$(not necessarily mutually orthogonal).

We then write

$$X[\phi \,] \, \chi_n = \lambda_n \, \chi_n \, , \tag{8.4}$$

and

$$\psi(x) = \sum a_n \, \chi_n(x). \tag{8.5}$$

While we could expand [22] $\bar{\psi}$ in terms of eigenfunctions of another operator $Y[\phi]$,we choose not to do so in the present work as it is not necessary,and expand

$$\bar{\psi}(x) = \sum \bar{b}_n \chi_n^\dagger(x) \tag{8.6}$$

and define

$$D\psi\ D\bar{\psi} = \prod_n da_n \prod_m d\bar{b}_m \tag{8.7}$$

Let us consider the problem of obtaining the one-loop expectation value of a fermion bilinear operator $O \equiv \bar{\psi}\ F[\partial,\phi(x)]\ \psi$ where F contains derivative operators, $\phi(x)$ and derivatives of ϕ. We understand this operator as

$$O \equiv \sum_n \sum_m \bar{b}_m\ a_n\ F_{mn}(x), \tag{8.8}$$

with

$$F_{mn}(x) \equiv \chi_m^{\dagger}(x)\ F\ [\partial,\phi(x)]\ \chi_n(x) \tag{8.9}$$

We shall then *define*, in a natural way,

$$O^{Reg}[a,\bar{b},\phi] \equiv \sum_n \sum_m b_m\ a_n\ F_{mn}(x)\ f(\lambda_n)\ g(\lambda_m), \tag{8.10}$$

where f and g are certain regularizing functions with $f(0)=g(0)=1$ and derivatives of f and g at $\lambda\to\infty$ are constrained to vanish sufficiently rapidly [22].

Then, we have

$$\left\langle O^{Reg} \right\rangle_{\phi} = \frac{\int \prod_n da_n \prod_m d\bar{b}_m\ O^{Reg}[a,\bar{b},\phi]\ e^S}{\int \prod_n da_n \prod_m d\bar{b}_m\ \ e^S}, \tag{8.11}$$

with S as

$$S \equiv \sum_n \sum_m \bar{b}_m\ (i\tilde{D} - m_o)_{mn}\ a_n\ . \tag{8.12}$$

$$(i\tilde{D} - m_o)_{mn} \equiv \int d^4x \, \chi_m^\dagger(x) \; (i\tilde{D} - m_o) \; \chi_n(x) .$$

Then using

$$\left\langle \bar{b}_m \, a_n \right\rangle = \{ \; (i\tilde{D} - m_o)^{-1} \; \}_{nm} \equiv \; G_{nm} \; , \tag{8.13}$$

we obtain

$$\left\langle \; O^{Reg} \; \right\rangle_\phi = \sum_m \sum_n \; g(\lambda_m) \; F_{mn}(x) \; f(\lambda_n) \; G_{nm} \; . \tag{8.14}$$

Expressed back in coordinate space this reads[2]

$$\left\langle \; O^{Reg} \right\rangle_\phi = \mathrm{Tr} \int d^4z \quad \delta^4(x-z) \; g \; (\overleftarrow{X}) F[\partial,\phi] \; f(X) \; G(x,z) , \tag{8.15}$$

with $G(x,z)$ the Fermion propagator in presence of ϕ field.

Generally $\langle O^{Reg} \rangle_\phi$ will depend on the regularizing mass parameter M. When expanded in powers of M, it may contain divergences having positive powers of M. We define renormalized operator $\langle O^R \rangle$ by simply subtracting these. (Note that ambiguities inherent in these subtractions do *not* generally account for *all* renormalization ambiguities. See, for instance, the two examples discussed in the next section). What we wish to show in the next

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[2]The relevant algebra is analogous to that found in chapter II [13]. One may also imagine obtaining it directly from the $\psi,\bar{\psi}$ dependent form for O, using $G(x,y) = \langle\psi(x) \; \bar{\psi}(y)\rangle$ in Eq. (8.15) and working backwards to Eq. (8.14).

two sections is that having adopted this convention, once and for all, for defining $\langle O^R \rangle$, *all* the ambiguities in $\langle O^R \rangle$ can be correlated, at one-loop level, with the arbitrariness in the choice of basis for defining the path-integral.

8.3 SIMPLE EXAMPLES

In this section we shall illustrate the point we wish to make in this chapter with the help of two examples, one a scalar operator and one a vector operator; The second operator, in addition, happens to be a finite operator so that the discussion in this case is completely independent of the question of subtraction of divergences.

Consider the operator[3]

$$O[\psi,\bar{\psi},\phi] = \bar{\psi}\,\frac{\delta S}{\delta \bar{\psi}} = \bar{\psi}(x)\ (i\tilde{D} - m_o)\ \psi(x) \tag{8.16}$$

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[3]In the dimensional regularization treatment, the expectation value of the equation of motion operator $\langle \bar{\psi}\ \delta S/\delta\bar{\psi} \rangle_\phi$ is zero when there are no external fermion sources [54]. Actually, as this operator contains pieces which are potentially divergent, setting this to zero is a particular way of renormalizing it. (For example, consider equation (8.20) below. Here $\langle O^{Reg} \rangle_\phi$ is formally $\sim \delta^4(0)$, which is zero in dimensional regularization [55]. However, when evaluated by Fujikawa method using field dependent regulators, it acquires different, non zero values, as seen in the evaluation of (8.20)). In Fujikawa's treatment, equation of motion operators acquire different values, and anomalies in his treatment are dependent on these [52,53].

As $O[\psi,\bar{\psi},\phi]$ is a dimension four operator, in evaluating $\langle O[\psi,\bar{\psi},\phi]\rangle$ in one loop approximation, primitive divergences will arise in a proper vertex with four or less ϕ lines. As a result with any suitable regularization and renormalization scheme, $\langle O^R\rangle$ will be ambiguous as indicated below. Let $\langle O_1^R\rangle$ and $\langle O_2^R\rangle$ represent $\langle O^R\rangle$ renormalized according to two different regularization and subtraction schemes. Then,

$$\left\langle O_1^R \right\rangle_\phi - \left\langle O_2^R \right\rangle_\phi = a\phi^2+b\phi^3+c\phi^4+d\phi\ \partial^2\phi+e\ \partial_\mu\phi\ \partial^\mu\phi + f\ \partial^2\phi^2$$

$$\equiv X_1[\phi;a,b,\text{——},f] \tag{8.17}$$

where $a,b,\text{——},f$ are finite constants of appropriate dimensions.

The question we are raising is whether we can understand completely the ambiguities in Eq.(8.17), *completely* as the arbitrariness available in the operator X used to define $D\psi$. We shall give a simple way to see that this is so. (There may be other constructions of X).

In Eq.(8.15) we set

$$F[\partial,\phi] = (i\tilde{D} - m_o) \ . \tag{8.18}$$

Choose $f = 1$ and use

$$(i\tilde{D}_x - m_o)\ G(x,z) = \delta^4(x-z) \tag{8.19}$$

to obtain that in the present case

$$\langle O^{Reg}\rangle_\phi = \mathrm{Tr}\int d^4z\ \delta^4(x-z)\ \mathcal{G}_M(\overleftarrow{X})\ \delta^4(x-z)$$

$$= \mathrm{Tr}[\ \delta^4(x-z)\mathcal{G}_M(\overleftarrow{X}\ )]_{x=z} \ . \tag{8.20}$$

Suppose we first choose $X=\tilde{D}$ and φ a function satisfying Fujikawa conditions. Then calling the operator so regularized, $<O_2^{Reg}>_\phi$,we find that $<O_2^{Reg}>_\phi$ has a divergence of the form of $(\alpha M^4 + \beta M^2\phi^2)$; so that

$$<O_2^{Reg}>_\phi - \alpha\, M^4 - \beta M^2\phi^2 = \text{finite} \equiv <O_2^R>_\phi \qquad (8.21)$$

This ,for us,defines a "standard" renormalization for <O>.We want to show that there is a choice of X in eq.(8.20) which will then reproduce all the ambiguities in Eq.(8.17).We show this by explicit construction.Let (with ξ to be determined)

$$X \equiv X_0 + \xi\ X_1 = \tilde{D}^4 + \xi\ X_1[\phi;\ a,b,—,f]\ . \qquad (8.22)$$

and $\tilde{\varphi}(x)$,a suitable rapidly decreasing function (as $x\to\infty$).

We consider

$$\left\langle O_1^{Reg}\right\rangle_\phi = \mathrm{Tr}\ [\ \delta^4(x-z)\ \tilde{\varphi}(\overleftarrow{X}/M^4)]\Big|_{x=z}\ . \qquad (8.23)$$

This is easily cast in the form (following Ref. 7):

$$\left\langle O_1^{Reg}\right\rangle_\phi = \mathrm{Tr}\int \frac{d^4k}{(2\pi)^4}\ 1.\ \tilde{\varphi}\left[\frac{(\overleftarrow{\tilde{D}} + i\not{k})^4 + \xi\ X_1[\phi]}{M^4}\right] \qquad (8.24)$$

Imagine setting $\xi = 0$ in the above expression.Then the finite part of right-hand side is independent of the function $\tilde{\varphi}$ (provided Fujikawa conditions are satisfied).The proof proceeds as in Ref.7.The divergent part is of the form $\alpha' M^4 + \beta' M^2\phi^2$. Thus $<O_1^R>_\phi$ reduces to that with the regulator $X = \tilde{D}$.ie,

$$\left\langle O_1^R \right\rangle_\phi \Big|_{X_1=0} = \left\langle O_2^R \right\rangle_\phi \; . \tag{8.25}$$

We want to know ,now,the contribution to the right hand side of Eq.(8.24) due to the presence of X_1.Carrying out the scaling $k \to Mk$,and $d^4k \to M^4 d^4k$ as usual and noting the presence of $1/M^4$ before $X_1[\phi]$,it is easily seen that contribution to Eq.(8.24) due to the presence of X_1 is solely,

$$Y \equiv 4\xi \left[\int \frac{d^4k}{(2\pi)^4} \quad \tilde{q}'(k^4) \right] X_1[\phi] \; , \tag{8.26}$$

where the prime as usual indicates differentiation with respect to argument.It is easily seen that

$$\begin{aligned} Y &= \xi \frac{2\pi^2}{(2\pi)^4} \quad [\tilde{q}\;(\infty) - \tilde{q}\;(0)]\; X_1[\phi] \\ &= -\frac{2\xi\pi^2}{(2\pi)^4} \quad X_1[\phi] \equiv X_1[\phi] = \text{finite} \; , \end{aligned} \tag{8.27}$$

with a proper numerical choice of ξ. Thus

$$\left\langle O_1^R \right\rangle_\phi = \left\langle O_2^R \right\rangle_\phi + X_1[\phi] = \text{finite} \; , \tag{8.28}$$

which is precisely the equation (8.17). This proves our assertion.(Finally,we note that while $\langle O_1^{Reg} \rangle_\phi$ has a divergence the ambiguity resulting from this divergence does not account for all the renormalization ambiguities given in Eq. (8.17)).

So far, we have given a particularly simple example in which construction of X was very straightforward.The simplicity of the example arose from the specific choice of $F[\partial,\phi] = (i\tilde{D} - m_o)$ and from $(i\tilde{D} - m_o)G(x,z) = \delta^4(x,z)$.Also helpful was the Lorentz

scalar nature of O.We shall now demonstrate our assertion in a somewhat complicated example of the Lorentz vector $O = \bar{\psi}\,\gamma_\mu\psi$. In this case ,

$$\left\langle O^{\mathrm{Reg}} \right\rangle_\phi = \mathrm{Tr} \int d^4z\ \delta^4(x-z)\ q(\tilde{X})\ \gamma_\mu\ f(X)\ G(x,z) \tag{8.29}$$

Let X be the dimension two operator[4]

$X = \tilde{D}^{\,2} + \alpha\phi^2 + \beta\gamma^\mu\partial_\mu\phi$. We also choose $q = 1$ and $f(x) = \exp\left[-\dfrac{X}{M^2}\right]$. (8.30)

One then has

$$\left\langle O^{\mathrm{Reg}} \right\rangle_\phi = \mathrm{Tr}\left\{ \gamma_\mu\ \exp[-X/M^2]\ G(x,z) \right\}\Bigg|_{x=z}$$

$$= \mathrm{Tr}\left[\gamma_\mu \int \frac{d^4k}{(2\pi)^4}\ \exp\left[- \frac{(\tilde{D}+i\not{k})^2+\alpha\phi^2+\beta\not{\partial}\phi}{M^2}\right] \frac{1}{-\not{k}-m_o} \right]$$

+ contributions from higher order terms in G.

(8.31)

After scaling $k \to Mk$, one has

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[4]In Eq. (8.30) we could parameterize X to bring out powers of q and hermiticity properties by replacing $\alpha \equiv q^2\alpha'$ and $\beta = i\beta' q$. We could also choose to preserve charge conjugation property by taking in Eq. (8.29) $q = f = \exp(-X/M^2)$. When this is done we find $\langle O^{\mathrm{Reg}}\rangle_\phi$ is real in Minkowski theory if α', β' and (iq) are regarded as real. Note that if (iq) is real, the Minkowski action is hermitian.

$$\left\langle O^{Reg}\right\rangle_{\phi} = M^3 \text{ Tr } \left[\gamma_\mu \int \frac{d^4k}{(2\pi)^4} e^{k^2} \exp\left[- \frac{2ik.\partial + 2ig\not{k}\phi}{M} - \frac{\tilde{D}^2 + \alpha\phi^2 + \beta\not{\partial}\phi}{M^2}\right] \times 1/(-\not{k} - m_o/M)\right] + \ldots\ldots\ldots \quad . \qquad (8.32)$$

A straightforward calculation shows that at $\alpha=\beta=0$, i.e with Fujikawa's usual regulator

$$\left\langle O_2^{Reg}\right\rangle = \left\langle O^{Reg}\right\rangle_{\phi} \Bigg|_{\substack{\alpha=0\\ \beta=0}} = \text{finite} \equiv \left\langle O_2^{R'}\right\rangle_{\phi} \quad . \qquad (8.33)$$

Focussing our attention on α and β dependent terms only,we find

$$\begin{aligned} <O_1^{Reg}>_{\phi} &= <O_2^R>_{\phi} + \xi\ (\alpha,\beta)\ \partial_\mu\phi^2 + \eta(\alpha,\beta) m_o \partial_\mu\phi \\ &\equiv <O_1^R>_{\phi}\ , \end{aligned} \qquad (8.34)$$

with

$$\begin{aligned} \xi(\alpha,\beta) &= - \frac{1}{16\pi^2}\left[\ i\alpha + 3ig\beta\ \right]\ , \\ \eta(\alpha,\beta) &= \frac{\beta}{4\pi^2} \quad . \end{aligned} \qquad (8.35)$$

Thus our choice of $X(\alpha,\beta)$ reproduces all the ambiguities in the renormalization of $<\bar{\psi}\ \gamma_\mu\psi\ >_\phi$.

What we have shown, with the help of these simple examples, is that all the ambiguities in $\langle O^R\rangle$ can be correlated with the arbitrariness in the choice of basis in the definition of path-integral and thus can interpreted as arising from it.

Finally we wish to make an important remark. While ambiguities in the definition of $<O^R[\psi\ \bar{\psi}\ \phi]>$ and the presence of

potential divergences will always go together, the ambiguities themselves have been seen to be arising from a probably independent source within the path-integral formulation, and thus have a significance quite apart from that associated with divergences. The ambiguities may be compared with, and are on the same footing as the overall phase of the wavefunctions in Quantum Mechanics. The latter can be chosen arbitrarily without altering physically observable results. In other words, a quantity depending on this phase cannot be an observable. Our discussion, in fact, then suggests that the proper vertices that have primitive divergence cannot be observables for a more fundamental reason than normally associated with.

To make our point of view very clear we make the following remark: Definition of the path-integral, like the phase of the wavefuntion, is always ambiguous irrespective of the question of divergences. This ambiguity may not have mattered, but the presence of potential divergences makes this ambiguity to have noticeable effects in calculation of Green's functions of operators.

8.4 FURTHER GENERALIZATIONS

In the previous section we explained our point of view regarding interpretration of renormalization counterterms with the help of renormalization of two simple operators. In this section we shall generalize the result to an arbitrary bilinear operator. Then, we shall make a few further comments.

First consider an arbitrary local fermion bilinear

operator of the form

$$O[\psi\ \bar{\psi}] = \bar{\psi}(x)\ \Gamma_i\ \partial_{\mu_1} \ldots\partial_{\mu_1}\ \psi(x)$$

$$+ \text{ Added Terms}, \qquad (n \geq 1) \tag{8.36}$$

where the last term in the above equation represents terms added in order to make the operator O traceless in the Lorentz indices $(\mu_1 \ldots \mu_n)$, and Γ_i is one of sixteen Dirac matrices $(\Gamma_i^2 \equiv 1)$. We shall consider the regularization and renormalization of this operator via the Fujikawa-type method discussed in the last section. Using Eq. (8.15) we can write

$$\langle\ O^{Reg}\ \rangle\phi \equiv \mathrm{Tr}\int d^4z\ \delta^4(x-z)\ \varphi(\overleftarrow{X})\Gamma_i\ \partial_{\mu_1} \ldots \partial_{\mu_n}\ f(X)\ G(x,z) \tag{8.37}$$

We have to show, now, that all ambiguities in the definition of $\langle\ O^R\ \rangle\phi$ can be accounted for by the arbitrariness in the choice of X. To show this, it is sufficient to give *one* choice of X which reproduces all renormalization ambiguities (though certainly many other choices would exist).

Let the renormalization ambiguities in $\langle\ O^R\ \rangle$ be represented by local finite polynomial $F^i_{\mu_1 \ldots \mu_n}[\phi]$ of degree (n+3): viz.

$$\langle\ O^R\ \rangle_1 - \langle\ O^R\ \rangle_2 = F^i_{\mu_1 \ldots \mu_n}[\phi,\ \alpha_1 \ldots \alpha_k], \tag{8.38}$$

where F depends on as many arbitrary coefficients (here k) as there are independent local monomials of appropriate Lorentz transformation property. Now consider

$$X \equiv (\tilde{D}^2)^{n+1} + \xi\ \Gamma_i\ \gamma^{\nu_1}\ F^i_{\nu_1 \ldots \nu_n}\ \partial^{\nu_2} \ldots \partial^{\nu_n}\ . \tag{8.39}$$

(there is no summation over i)

In Eq. (8.37) choose $q = 1$ and $f = e^{\frac{-X}{M^{2n+2}}}$. Then

$$\langle O^{Reg} \rangle\phi = \mathrm{Tr}\int d^4z\, \delta^4(x-z)\, \Gamma_i\, \partial_{\mu_1} \cdots \partial_{\mu_n}\, e^{\left[-\frac{X}{M^{2n+2}}\right]}\, G(x,z)$$

$$= \mathrm{Tr}\left\{ \Gamma_i\, \partial_{\mu_1} \cdots \partial_{\mu_n}\, e^{\left[-\frac{X}{M^{2n+2}}\right]}\, G(x,z) \right\}\Bigg|_{x = z} . \quad (8.40)$$

Expanding $G(x,z)$ as

$$G(x,z) = G_o(x,z) - ig\int G_o\,(x,y)\,\phi(y)\,G_o\,(y,z)\,d^4y + \ldots, \quad (8.41)$$

and further using

$$G_o(x,z) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik.(x-z)}\, \frac{1}{-\not{k} - m_o}, \quad (8.42)$$

we obtain, after using $\partial^x_\mu\, e^{ik.(x-z)} = e^{ik.(x-z)}\, (\partial^x_\mu + ik_\mu)$,

$$\langle O^{Reg} \rangle\phi \equiv \mathrm{Tr}\Bigg\{\Gamma_i (\partial_{\mu_1} + ik_{\mu_1}) \ldots (\partial_{\mu_n} + ik_{\mu_n})\, \exp\Bigg\{- \frac{[(\tilde{D} + i\not{k})^2]^{n+1}}{M^{2n+2}}$$

$$+ \frac{\xi\, \Gamma_i\, \gamma^{\nu_1}\, F^i_{\nu_1 \ldots \nu_n}\, (\partial^{\nu_2} + ik^{\nu_2}) \ldots (\partial^{\nu_n} + ik^{\nu_n})}{M^{2n+2}}\Bigg\}$$

$$\times\ \frac{-1}{\not{k} + m_o}\Bigg\} + \text{contributions from higher order terms in } G. \quad (8.43)$$

Now we define $\langle O_2^{Reg} \rangle = \langle O^{Reg} \rangle \Big|_{\alpha_i \to 0}$ i.e., $\langle O_2^{Reg} \rangle$ is obtained by replacing F by zero. Hence,

$$\langle O_1^{Reg} \rangle_\phi - \langle O_2^{Reg} \rangle_\phi = \text{terms dependent on } \alpha_1 \ldots \alpha_k .$$

We now show that there is only one term in Eq. (8.43) depending on $\alpha_1 \ldots \alpha_k$ as $M \longrightarrow \infty$. To see this first we rescale $k \longrightarrow Mk$. Consider the M power of the leading term in M depending on F :

$$= i(-1)^n \int M^4 d^4k (Mk_{\mu_1}) \ldots (Mk_{\mu_n}) \frac{1}{M^{2n+2}} \mathrm{Tr}\Big\{ \Gamma_i (\xi \Gamma_i) \gamma^{\nu_1} F^i_{\nu_1 \ldots \nu_n} (Mk^{\nu_2}) \ldots$$

$$(Mk^{\nu_n}) \frac{1}{M\not{k} + m_o} \Big\} \tag{8.44}$$

$$\sim M^{4+n} \frac{1}{M^{2n+2}} M^{n-1} \frac{1}{M} = O(M^0) .$$

All other F-dependent terms vanish as $M \quad \infty$. We evaluate the term in Eq.(8.44) using $\mathrm{tr}\,(\Gamma_i^2 \gamma^{\nu_1} k) = \mathrm{tr}\,(\gamma^{\nu_1} k) = 4k^{\nu_1}$, tracelessness of $F^i_{\nu_1 \ldots \nu_n}$ in any pair of indices $\nu_1 \ldots \nu_n$, symmetry of F and

$$\int d^4k \frac{k_{\mu_1} \ldots k_{\mu_n} \; k^{\nu_1} \ldots k^{\nu_n}}{k^2} e^{-(-k^2)^{n+1}} = C \Big\{ \delta^{\nu_1}_{\mu_1} \ldots \delta^{\nu_n}_{\mu_n} + \text{permutations of } \{ \nu_1 \ldots \nu_n \} \Big\}$$

+ terms which do not contribute due to tracelessness of F.

(C is the appropriate constant).

We find, as $M \longrightarrow \infty$

$$\langle O_1^{Reg} \rangle_\phi - \langle O_2^{Reg} \rangle_\phi = \xi \; (4iC(-1)^n) \; F_{\mu_1 \ldots \mu_n}[\phi] \tag{8.45}$$

$$= \text{finite.}$$

Hence using definition of $\langle O^R \rangle$ and choosing $\xi = [4iC(-1)^n]^{-1}$

We find

$$\langle O_1^R \rangle - \langle O_2^R \rangle = F_{\mu_1 \ldots \mu_n}[\phi, \alpha_1 \ldots \alpha_k] , \tag{8.46}$$

which is precisely Eq. (8.38).

The operator in Eq. (8.36) is definied for $n \geq 1$. Consider the $n = 0$ case.

Here,

$$O[\psi \bar{\psi}] = \bar{\psi} \Gamma_i \psi . \tag{8.47}$$

A choice of X reproducing the renormalization ambiguities for this operator is

$$X \doteq \tilde{D}^4 + \xi \Gamma_i F^i \not{\phi} . \quad \text{(no summation over i)} \tag{8.48}$$

$\left(\langle O^{Reg} \rangle_\phi\right.$ is given by Eq. (8.15) with $F = \Gamma_i$, $f(x) = e^{-\frac{X}{M^4}}$ and $q = 1\Big)$

Now any bilinear opeator $\bar{\psi}H(\partial, \phi, \partial\phi, \ldots.)\psi$ can be represented as

$$\bar{\psi}H(\partial, \phi, \partial\phi, \ldots.)\psi = \sum a_n H_n[\phi, \partial\phi, \ldots.]\bar{\psi} \Gamma^{(n)} \partial_{\mu_1} \ldots \partial_{\mu_n} \psi , \tag{8.49}$$

and hence the result for the operators of the form (8.36) and (8.47) can be generalized to any such operator in one loop order.

Suppose now we consider an operator of the type

$$\tilde{O}[\psi \bar{\psi}] = \bar{\psi}(x) \overleftarrow{\partial}_{\mu_1} \ldots \overleftarrow{\partial}_{\mu_n} \Gamma_i \partial_{\mu_{m+1}} \ldots \partial_{\mu_n} \psi(x) \quad (n \geq 1). \tag{8.50}$$

Such an operator can always be written in terms of $\bar{\psi}(x)\partial \ldots \partial \psi(x)$ and its total derivatives:

$$\tilde{O}[\psi\ \bar{\psi}]=(-)^{m}\ \bar{\psi}\partial_{\mu_1}\cdots\partial_{\mu_m}\partial_{\mu_{m+1}}\cdots\cdots\partial_{\mu_n}\ \psi + a \sum_{s=1}^{m}\partial_{\mu_s}\left\{\bar{\psi}\prod_{\substack{r=1\\(r\neq s)}}^{m}\partial_{\mu_r}\partial_{\mu_{m+1}}\cdots\partial_{\mu_n}\psi\right\}$$

$$+ \cdots\cdots\cdots \quad . \tag{8.51}$$

Noticing that for an operator O and its total derivatives

$$\langle\ (\ \partial_{\mu}\ \cdots\ \partial_{\nu}\ \hat{O}\ )^{R}\ \rangle\ =\ \partial_{\mu}\cdots\partial_{\nu}\ \langle\ O^{R}\ \rangle\ , \tag{8.52}$$

renormalization ambiguities in operators of Eq. (8.50) can be understood in terms of the ambiguities of the operators of the form given in Eq. (8.36). By the same token, the operators of the form $\bar{\psi}\tilde{H}\ (\overleftarrow{\partial},\ \partial,\ \partial\phi,\ \phi,\ldots)\psi$ can be written in terms of $O[\psi.\bar{\psi}]$ and $\tilde{O}[\psi,\bar{\psi}]$, considered already, by an expansion similar to that in Eq. (8.49) and thus any renormalization ambiguity in renormalization of such an operator is reproducible by varying the local operator X chosen to define the path-integral.

We conclude by making a couple of comments. As has been mentioned before, the choice of operator X is not unique. Take the example of the operator we have already considered, viz., $O = \bar{\psi}\gamma_{\mu}\psi$. The renormalization ambiguities in $\langle\ O^{R}\ \rangle$ have already been seen to be reproduced by the dimension two operator given in Eq. (8.30). Another choice that would work as well is the dimension four operator of Eq. (8.48) with $\Gamma \equiv \gamma_{\mu}$.

Our discussion of renormalization ambiguities in the renormalization of operators has been restricted to one-loop level. Actually, regularization and renormalization in Fujikawa's Path-Integral formulation has rarely been carried beyond one-loop

approximation [52]. Until this is done we cannot comment on the validity of the view presented here beyond one-loop approximation. (It may be noted that operators quartic (or higher) in fermion fields contain at least two loops in their expectation values).

8.5 CONCLUSION

In this chapter we discussed the renormalization of bilinear composite operators in path-integral framework at one-loop level in the setting of a Yukawa type theory. We have shown that all ambiguities in their renormalization can be understood within the path-integral approach as arising from the arbitrariness in the choice of basis for the definition of the path-integral. We also pointed out that just as the quantum mechanical wavefunction is defined only upto a phase factor which can be chosen without altering the physically observable results, the definition of the path-integral is ambiguous due to the freedom in the choice of basis to define the path-integral. This ambiguity may not have mattered, but due to the presence of potential divergences in calculation of Green's functions of operators, it has noticeable effects.

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APPENDIX A

In this appendix we shall present a general procedure for dealing with the terms in Eq. (2.48). To this end, we shall choose y and $z \equiv x-y$ as independent variables. Then the terms in the Eq. (2.48) become, upto an overall factor,

$$\Big[\int d^2y\ d^2z\ d^2k\ d^2k' e^{i(k+k').z} \exp\left(\frac{k^2}{M^2}\right) \exp\left\{-\frac{\not{D}_a^2}{M^2} - \frac{2ik.D_a}{M^2}\right\}_{y+z} .1$$

$$\times\ \frac{\not{k} - m_o}{k^2 - m_o^2}\ \alpha(y)\gamma_5 \exp\left(\frac{k'^2}{M^2}\right) \exp\left\{-\frac{\not{D}_{ay}^2}{M^2} + \frac{2ik'.D_{ay}}{M^2}\right\} \not{A}(y)$$

$$+ \text{ the second term } \Big]. \qquad \text{(A.1)}$$

First, we shall present a general procedure for dealing with the exponential, $\exp\left\{-\frac{\not{D}_a^2}{M^2} - \frac{2ik.D_a}{M^2}\right\}_{(y+z)}$. The first term, 1, in its expansion, of course, contributes and has been partly dealt with in Sec.(2.5), and will be partly dealt with later in this appendix. We shall deal with the remaining terms here. Consider for concreteness the term $-\frac{2ieak.A(y+z)}{M^2}$. We expand $A_\sigma(y+z)$ around z=0 and express

$$A_\sigma(y+z) = A_\sigma(y) + z^\tau \frac{\partial}{\partial y\tau} A_\sigma(y) + \ldots\ldots . \qquad \text{(A.2)}$$

Then this term, with $A_\sigma(y+z)$ replaced by $A_\sigma(y)$, gives the contribution in the first term in (A.1):

$$-\frac{2iea}{M^2}\int d^2y\ d^2z\ d^2k\ d^2k'\ e^{i(k+k').z} \exp\left(\frac{k^2}{M^2}\right) k.A(y) \frac{\not{k} - m_o}{k^2 - m_o^2}\ \alpha(y)\gamma_5$$

$$\times \exp\left(\frac{k'^2}{M^2}\right) \exp\left\{-\frac{\not{D}_{ay}^2}{M^2} + \frac{2i\ k'.D_{ay}}{M^2}\right\} \not{A}(y).$$

Now the z-integration can be carried out explicitly yielding $\delta^2(k+k')$. Then the k'-integration can be carried out. This results in the expression

$$\frac{1}{M^2}\int d^2y\ d^2k\ \exp\left(\frac{2k^2}{M^2}\right) k.A(y)\ \frac{\not{k} - m_o}{k^2 - m_o^2}\ \alpha(y)\gamma_5\ \exp\left\{\frac{-\not{D}_{ay}^2}{M^2} - \frac{2ik.D_{ay}}{M^2}\right\} \not{A}(y).$$

We now scale k = Mk" Then it is easy to see that as $M \to \infty$, the only term in $\exp\left\{\frac{-\not{D}_{ay}^2}{M^2} - \frac{2ik".D_{ay}}{M}\right\}$ that will contribute, is 1. This contribution can be easily evaluated and be shown to cancel with an identical contribution from "the second term" in (A.1).

The same procedure is to be applied to the remaining terms in the expansion in (A.2). It is easy to show that the z and k' integrations can be carried out and the terms do not survive as $M \to \infty$.

In an identical manner, the remaining terms in the expansion of $\exp\left\{-\frac{\not{D}_a^2}{M^2} - \frac{2ik.D_a}{M^2}\right\}_{(y+z)}$ are dealt with and shown not to contribute as $M \to \infty$.

We, now deal with remaining term obtained by replacing $\exp\left\{-\frac{\not{D}_a^2}{M^2} - \frac{2ik.D_a}{M^2}\right\}_{(y+z)}$ by 1 in (2.48). We can now perform z-integration yielding $\delta^2(k+k')$ and then the k' integration. The result, upto an overall factor, is

$$\int d^2y\ d^2k\ \exp\left\{\frac{2k^2}{M^2}\right\} \frac{-1}{\not{k} + m_o}\ \alpha(y)\gamma_5\ \exp\left\{\frac{-\not{D}_{ay}^2}{M^2} - \frac{2ik.\ D_{ay}}{M^2}\right\} \not{A}(y)$$

$$+ \text{ the second term.}$$

We now scale $k \longrightarrow Mk$ and obtain

$$M \int d^2y\, d^2k \exp\left\{2k^2\right\} \frac{-1}{\not{k} + m_o/M}\, \alpha(y)\gamma_5 \exp\left\{ \frac{-\not{D}^2_{ay}}{M^2} - \frac{2ik.D_{ay}}{M} \right\} \not{A}(y)$$

+ the second term.

As $M \longrightarrow \infty$, only the term $\frac{-2ik.D_{ay}}{M}$ contributes from the exponential. We have already dealt with the term $\frac{-2ik.\partial_y}{M}$ in Sec (2.5). The remaining contribution is

$$M \int d^2y\, d^2k \exp\left\{2k^2\right\} \frac{-1}{\not{k} + \frac{m_o}{M}}\, \alpha(y)\gamma_5 \left(- \frac{2ik.A}{M}\right) \not{A}(y)\ (iea)$$

+ the second term. (A.3)

This contribution is easily shown to be zero as a consequence of the cancellation of terms in (A.3) using $\gamma_5 \not{A} + \not{A}\gamma_5 = 0$.

APPENDIX B

In this appendix, we shall show that there is no contribution coming from terms other than the free Green's function in Eq. (2.39), when the expansion (2.45) for G(x,y) is used.

We first consider the term arising from G_1 of Eq. (2.45). Omitting an overall constant it yields

$$\lim_{M\to\infty} \mathrm{Tr}\Big[\int d^2x d^2y \quad e^{-\not{D}^2_{ax}/M^2} \left\{\int d^2z \; G_o(x,z) \; \not{A}(z) G_o(z,y)\right\} \alpha(y) \; \gamma_5$$

$$\times \int d^2k' \quad e^{-\not{D}^2_{ay}/M^2} \; e^{ik'\cdot(x-y)} \quad \not{A}(y)$$

$$+ \int d^2x \; d^2y \quad e^{-\not{D}^2_{ax}/M^2} \left\{\int d^2z \; G_o(x,z) \; \not{A}(z) \; G_o(x,y)\right\} \not{A}(y)$$

$$\times \int d^2k' \quad e^{-\not{D}^2_{ay}/M^2} \; e^{ik'\cdot(x-y)} \; \alpha(y)\gamma_5\Big] \;. \qquad (B.1)$$

In the above, we express

$$G_o(x,z) \quad = \quad - \int \frac{d^2k_1}{(2\pi)^2} \; \frac{\not{k}_1 - m_o}{k_1^2 - m_o^2} \; e^{ik_1\cdot(x-z)} \;,$$

$$(B.2)$$

$$G_o(z,y) \quad = - \int \frac{d^2k_2}{(2\pi)^2} \; \frac{\not{k}_2 - m_o}{k_2^2 - m_o^2} \; e^{ik_2\cdot(z-y)} \;.$$

We then carry out the z-integration. Then we change the variables to $k_1 = k_1$ and $q = k_1 - Mk_2$ in momentum space, and to y=y and w = x-y in coordinate space. A little simplification allows us to express (B.1), again omitting an overall factor, as

$$\lim_{M\to\infty} \mathrm{Tr}\left\{\int d^2y\ d^2w\ d^2k_1\ d^2q\ \ d^2k'\ e^{iq.y}\ e^{iw.(k'+k_1)}\ e^{k_1^2/M^2}\ e^{k'^2/M^2}\right.$$

$$\times\ \left[\exp\left\{-\frac{\not{D}_a^2}{M^2} - \frac{2i}{M^2}\,k_1.D_a\right\}_{y+w}.1\right] \frac{\not{k}_1 - m_o}{k_1^2 - m_o^2}\ \tilde{\not{A}}\ (q)$$

$$\times\ \frac{\not{k}_1 - \not{q} - m_o}{(k_1-q)^2 - m_o^2}\ \alpha(y)\gamma_5\ \exp\left\{-\frac{\not{D}_{ay}^2}{M^2} + \frac{2i}{M^2}\,k'.D_{ay}\right\}\not{A}(y)$$

$$\left.+\ \text{other term}\right\}. \tag{B.3}$$

We now rescale $k' \longrightarrow Mk'$, $k_1 \longrightarrow Mk_1$, $w \longrightarrow w/M$ to obtain

$$\lim_{M\to\infty} \mathrm{Tr}\int d^2y\ d^2w\ d^2k_1\ d^2q\ d^2k'\ e^{iq.y}\ e^{iw(k'+k_1)}\ e^{k_1^2}\ e^{k'^2}$$

$$\times\ \left[\exp\left\{-\frac{\not{D}_a^2}{M^2} - \frac{2i}{M}\,k_1.D_a\right\}_{y+w/M}.1\right] \frac{\not{k}_1 - m_o/M}{k_1^2 - m_o^2/M^2}\ \tilde{\not{A}}\ (q)$$

$$\times\ \frac{\not{k}_1 - \not{q}/M - m_{o/M}}{\left(k_1 - q/M\right)^2 - m_{o/M^2}^2}\ \ \alpha(y)\gamma_5\ \exp\left\{-\frac{\not{D}_{ay}^2}{M^2} + \frac{2ik'.D_{ay}}{M}\right\}\not{A}(y)$$

$$+\ \text{other term} \tag{B.4}$$

If necessary, $\exp\left\{-\frac{\not{D}_a^2}{M^2} - \frac{2ik_1.D_a}{M}\right\}_{y+w/M}.1 = f(y+w/M)$ can be expanded in a Taylor series in w/M and the w and then k' integrations can be carried out. Using this procedure, or otherwise, it is evident from (B.4) that the only terms that can survive as $M\to\infty$ are the ones in which the exponential operator factors $\exp\left\{-\frac{\not{D}_a^2}{M^2} - \frac{2ik_1.D_a}{M}\right\}$ and $\exp\left\{\frac{\not{D}_{ay}^2}{M^2} + \frac{2ik'.D_{ay}}{M}\right\}$ are replaced by one. These contributions are

$$\lim_{M\to\infty} \int d^2y\, d^2w\, d^2k_1\, d^2q\, d^2k'\, e^{iq\cdot y}\, e^{iw\cdot(k'+k_1)}\, e^{k_1^2}\, e^{k'^2}$$

$$\frac{\not{k}_1 - m_o/M}{k_1^2 - m_o^2/M^2}\, \not{A}\,(q)\, \frac{\not{k}_1 - \not{q}/M - m_o/M}{(k_1 - q/M)^2 - m_o^2/M^2} \left[\alpha(y)\gamma_5\, \not{A}(y) + \not{A}(y)\alpha(y)\gamma_5\right], \tag{B.5}$$

which is regularized and vanishes because the term in the last square bracket vanishes.

The contributions from G_2, G_3, ... can be dealt with more easily. These contributions are finite without regulators. So we let $M \longrightarrow \infty$ inside the integrals. Then the corresponding contributions from the two terms in Eq. (2.39) will add up to zero when $\gamma_5 \not{A} + \not{A}\, \gamma_5 = 0$ is used inside the integrals.

APPENDIX C

In this appendix we shall deal with the terms on the right-hand side of Eq. (2.32) except the first, and show that they do not contribute. First, consider

$$\sum_p \sum_n \frac{z^M_{pn} \, (\not{A})_{np}}{(i\lambda_p - m_o)\,(i\lambda_n - m_o)}, \tag{C.1}$$

where z^M_{pn} is given in Eq. (2.33) as

$$z^M_{pn} = i \sum_s \left[(\alpha\gamma_5)_{ps} \, (\not{A})_{sn} + (\alpha\gamma_5)_{sn} \, (\not{A})_{ps}\right] e^{-\lambda^2_s/M^2 - \lambda^2_p/M^2} \tag{C.2}$$

When (C.2) is used in (C.1) and simplifications are made using Eq.(2.37), we obtain, for (C.1),

$$\sum_p \sum_n \frac{z^M_{pn} \, (\not{A})_{np}}{(i\lambda_p - m_o)(i\lambda_n - m_o)} = i \, \mathrm{tr}\Big[\int d^2x \; d^2y \; d^2z \; G(x,z) \; \not{A}(z) G_M(z,y) \; \gamma_5 \alpha(y)$$

$$\times \; e^{-\not{\partial}^2_{ay}/M^2} \; \delta^2(x-y) \; \not{A}(y)\Big]$$

$$+ i \; \mathrm{tr}\Big[\int d^2x \; d^2y \; d^2z \; \gamma_5 \; G(x,z) \; \not{A}(z) \; G_M(z,y) \; \not{A}(y) \; e^{-\not{\partial}^2_{ay}/M^2} \delta^2(x-y) \alpha(y)\Big]. \tag{C.3}$$

First, consider the contribution to the right-hand side of Eq. (C.3) coming from the free Green's function. It is

$$i\ \mathrm{tr}\left[\int d^2x\ d^2y\ d^2z \left[G_o(x,z)\ \not{A}(z)\ G_{oM}(z,y)\gamma_5\ \alpha(y)\ e^{-\not{D}^2_{ay}/M^2}\ \delta^2(x-y)\ \not{A}(y)\right.\right.$$

$$\left.\left. +\ \gamma_5\ G_o(x,z)\ \not{A}(z)\ G_{oM}\ (z,y)\ \not{A}(y)\ e^{-\not{D}^2_{ay}/M^2}\ \delta^2(x-y)\ \alpha(y)\right]\right] . \tag{C.4}$$

This contribution is almost the same in form as the contribution from G_1 dealt with in Appendix B (See Eq. B.1) except that the second free Green's function is regularized rather than the first. $\left(e^{-\not{D}^2_{ax}/M^2}\ G_o(x,y) = G_{oM}(x,y)\right)$. This does not alter much the evaluation of (C.4) and is readily shown to be zero by a process similar to that in appendix B.

The contributions (C.3) coming from higher order terms in G are finite without regularization and vanish between the two terms using $\not{A}\gamma_5 + \gamma_5\not{A} = 0$.

In a similar manner, the remaining contributions to the right-hand side of Eq.(2.32) are finite without regularization, and when regularization is removed, $Z_{pq} = 0$, as seen using $\gamma_5\not{A} + \not{A}\ \gamma_5 = 0$.

APPENDIX D

In this appendix, we shall give some technical details of the contribution of the free Green's function to the right-hand side of Eq. (4.21).

Consider the first term on the right-hand side of Eq. (4.21). This term, using Eqs. (4.26) and (4.27), becomes

$$- ie(1-a) \int \frac{d^4x\ d^4y\ d^4k\ d^4k'}{(2\pi)^8}\ e^{i(k-k').(x-y)}\ \exp\left[\frac{k^2+k'^2}{M^2}\right]$$

$$x\ \mathrm{Tr}\left\{\exp\left[-\ \frac{\not{D}^2_{ax}}{M^2} - \frac{2ik.D_{ax}}{M^2}\right].1\ \frac{\not{k}-m_o}{k^2-m_o^2}\ \alpha(y)\gamma_5\ \exp\left[-\ \frac{\not{D}^2_{ay}}{M^2} - \frac{2ik'.D_{ay}}{M^2}\right]\not{A}(y)\right\}\ .$$

We expand $\exp\left[-\ \frac{\not{D}^2_{ax}}{M^2} - \frac{2ik.D_{ax}}{M^2}\right].1$ around x = y in a Taylor series. We write x = y+z and change variables of interation to y and z. Then the z-integration can be done which yields $\delta^4(k-k')$ or its derivatives. The k' integration can also be done, leaving only one momentum integration over k. We scale this k, by M. One then has to count the powers of M coming from various sources and only keep those terms which have no net negative power of M. To this end we note that a) d^4k yields a factor of M^4, propagator yields a factor of M^{-1} and each derivative of $\delta^4(k-k')$ would yield M^{-1}; b) In order to get a nonzero trace with γ_5, there must be at least four γ-matrices; c) The resulting integrand must be an even function of k. These considerations limit the (possibly) contributing terms to the following :

(i) $\frac{\not{D}^2_{ax}}{M^2}$ from the first exponential and $\frac{k'.D_{ay}}{M^2}$ term from the second;

(ii) $\frac{k.A}{M}$ from the first exponential and $\frac{\not{\partial}^2_{ay}}{M^2}$ term from the second;

(iii) $\partial_\mu \left(\frac{\not{\partial}^2_{ax}}{M^2}\right)$ from the first exponential and 1 from the second;

(iv) $\frac{\not{\partial}^2_{ax}}{M^2} \cdot \frac{k.D_{ax}}{M^2}$ type term from the first exponential and 1 from the second;

(v) 1 from the first exponential and $\frac{(\not{\partial}^2_{ay})\ (k'.D_{ay})}{M^4}$ type term from the second.

Of these, contribution (ii) is easily seen to be proportional to $\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}\ A_\lambda A_\sigma$ which vanishes in the Abelian case. Contributions (iii) and (iv) when summed over the two terms on the right-hand side of Eq. (4.21) are seen to vanish because of $\gamma_5\not{A} + \not{A}\gamma_5 = 0$ without the need of calculation.

Contribution (i) and (v) are both numerically equal and they together lead to the result of Eq. (4.28).

APPENDIX E

In this appendix, we shall deal with the contribution from G_1 term in the Green's function (see Eq. 4.24) to the chiral anomaly, given by Eqns. (4.29) and (4.31). The contribution from G_1 to the first term on the right-hand side of Eq. (4.21) is,

$$ie^2 a(1-a)\int d^4x\, d^4y \exp\left[-\frac{\not{D}^2_{ax}}{M^2}\right]\int d^4z\, G_o(x,z)\, \not{A}(z) G_o(z,y)\, \alpha(y)\gamma_5$$
$$\exp\left[-\frac{\not{D}^2_{ay}}{M^2}\right]\delta^4(x-y)\not{A}(y) \quad . \qquad (E.1)$$

Using Eqs. (4.26) and (4.27) in Eq. (E.1), we obtain, after some simplification,

$$ie^2 a(1-a)\int dx\, dy\, dk_1 dk_2 dk \exp\{ik_1.x - ik_2.y + ik.(y-x)\} \exp\left[\frac{-k_1^2+k^2}{M^2}\right]$$
$$\times\ \mathrm{Tr}\left\{\exp\left[-\frac{\not{D}^2_{ax}}{M^2} - \frac{2ik_1.D_{ax}}{M^2}\right]\frac{\not{k}_1-m_o}{k_1^2-m_o^2}\ \tilde{\not{A}}(k_1-k_2)\right.$$
$$\left.\times\ \frac{\not{k}_2-m_o}{k_2^2-m_o^2}\,\alpha(y)\ \gamma_5 \exp\left[-\frac{\not{D}^2_{ay}}{M^2} - \frac{2ik.D_{ay}}{M^2}\right]\not{A}(y)\right\} . \qquad (E.2)$$

We carry out a change of variables from x, y to w = (y-x) and y. As in Appendix D, we expand $\exp\left[-\frac{\not{D}^2_{ax}}{M^2} - \frac{2ik_1.D_{ax}}{M^2}\right].1$ in powers of ω in a Taylor series around x = y and then carry out the ω integration. This gives $\delta(k_1-k)$. Then the k_1 integration can be done. A further change of variables $k = \frac{p+q}{2}$, $k_2 = \frac{p-q}{2}$ is carried out. Then a rescaling $p \longrightarrow Mp$ is done keeping q fixed. The power counting used in Appendix D together with identities (i) $\mathrm{Tr}\{\gamma_5\ \not{k}\ \not{A}\ \not{k}\ \not{A}\ \} = 0$,

(ii) $\mathrm{Tr}\{\gamma_5 \sigma_{\mu\nu} \not{k} \not{A} \not{k} \not{A}\} = 0$ and (iii) $\mathrm{Tr}\{\gamma_5 \gamma_\alpha \not{A} \not{k} \not{A}\} = 0$, lead us to the only surviving term which comes from the first term 1 in the expansion of $\left[-\frac{\not{D}^2_{ax}}{M^2} - \frac{2ik.D_{ax}}{M^2}\right]$, and $\frac{k.D_{ay}}{M}$ term from $\exp\left[-\frac{\not{D}^2_{ay}}{M^2} - \frac{2ik.D_{ay}}{M^2}\right]$. The net result for this contribution is stated in Eq. (4.29).

Next, we shall come to the contribution $A_3(x)$ of the fourth term on the right hand side of Eq. (4.19). This contribution can be cast in the form

$$ie^2(1-a)^2\int dxdydz\; G(x,z)\; \not{A}(z)\; G(z,y)\; \exp\left[-\frac{\not{D}^2_{ay}}{M^2}\right]\alpha(y)\gamma_5 \exp\left[-\frac{\not{D}^2_{ay}}{M^2}\right] \delta^4(y-x)\; \not{A}(y)$$

$$+ \text{ a similar term} \tag{E.3}$$

We note that this term is very similar to the expression (E.1) except that $G_o \longrightarrow G$ and location of one of the regularizing exponentials is changed. Only the free Green's function contributes in (E.3) and the calculation is similar to that of the expression (E.1). The result is stated in Eq. (4.31).

APPENDIX F

In this appendix, we shall briefly sketch the calculation of the right hand side of Eq. (4.53). Firstly, however, we shall explain as to why the multiple commutator terms in Eq. (4.50) do not contribute. As is evident in many calculations using Fujikawa techniques, one would encounter

$$M^{-2n-2}\,[\not{D}_a^2,\ [\not{D}_a^2\ \cdots\cdots\ [\not{D}_a^2,\ \not{D}_a\not{\partial}\alpha+\not{\partial}\alpha\not{D}_a]\ldots]\ e^{ikx}$$
$$\longleftarrow \text{ n commutators } \longrightarrow$$

$$=e^{ik.x}M^{-2n-2}[(\not{D}_a+i\not{k})^2,[(\not{D}_a+i\not{k})^2,\ldots[(\not{D}_a+i\not{k})^2,\ (\not{D}_a+i\not{k})\not{\partial}\alpha+\not{\partial}\alpha(\not{D}_a+i\not{k})]..].$$

We note that the k^2 term in each of $(\not{D}_a+i\not{k})^2$ does not contribute to the commutator. As such, when the scaling $k \rightarrow Mk$ is carried out, the above expression is

$$= e^{ikM.x}\ M^{-2n-2}[\not{D}_a^2+2iMk.D_a,[(\not{D}_a^2+2iMk.D_a,\ldots,[(\not{D}_a^2+2iMk.D_a,\ldots.]..].$$

The leading term (in powers of M) is of $O(M^{-n-2})$ and thus the power of M associated with a nested nth-order commutator goes on decreasing as n increases. A simple power counting then shows that for $n > 2$, the contribution to Eq. (4.50) vanishes as $M \rightarrow \infty$. For $n = 2$, the only term that could contribute in the limit $M \rightarrow \infty$ vanishes as $\mathrm{tr}\,(\gamma_5\not{a}\not{b}) = 0$.

Now, consider one contribution to Eq. (4.53), it contains

$$M^{-2}\ [\not{D}_{ax}\not{\partial}\alpha+\not{\partial}\alpha\not{D}_{ax}]\ e^{ik.x}$$

$$= e^{ik.x}\ M^{-2}\ [(\not{D}_{ax}+i\not{k})\ \not{\partial}\alpha+\not{\partial}\alpha\ (\not{D}_{ax}+i\not{k})]$$

$$= e^{ik.x}\ M^{-2}\ [\not{\partial}\ \not{\partial}\alpha+\not{\partial}\alpha\not{\partial} + (iea\not{A} + i\not{k})\not{\partial}\alpha + \not{\partial}\alpha(iea\not{A} + i\not{k})]$$

$$= e^{ik.x}\ M^{-2}\ [\partial^2\alpha + 2\partial^{\mu}\alpha\ \partial_{\mu} + 2(ieaA + ik).\partial\alpha]. \tag{F.1}$$

After rescaling $k \longrightarrow Mk$, the square bracket becomes

$$\frac{\partial^2\alpha + 2\partial^\mu\alpha\, \partial_\mu + 2ieaA.\partial\alpha}{M^2} + \frac{2ik.\partial\alpha}{M} \,. \tag{F.2}$$

The above expression contains no γ-matrices. As seen earlier in Appendix D, a nonzero contribution is possible when γ_5 is accompanied by at least four γ-matrices. A simple power counting and counting of γ-matrices shows that only the second term in (F.2) can contribute. In a similar manner, the term containing B′ in Eq. (4.53) can be dealt with.

The contributions come both from the free Green's function G_o and also from G_1. The result is stated in Eq. (4.55).

Next consider the contribution to Eq. (4.53) arising from [B,A] type term. This term contains

$$M^{-4}\,[\not{D}^2_{ax},\ \not{D}_a \not{\partial}\alpha + \not{\partial}\alpha\, \not{D}_a]\, e^{ik.x}$$

$$= e^{ik.x}\, M^{-4}\, [(\not{D}_a + i\not{k})^2,\ (\not{D}_a + i\not{k})\, \not{\partial}\alpha + \not{\partial}\alpha\, (\not{D}_a + i\not{k})]$$

$$= e^{ik.x}\, M^{-4}\, [\not{D}^2_a + 2ik.D_a\ ,\ \partial^2\alpha + 2\partial^\mu\alpha\, \partial_\mu + 2(ieaA + ik).\partial\alpha].$$

As $M \longrightarrow kM$, this goes into

$$= e^{ikM.x}\, M^{-2}\, [2ik.D_a,\ 2ik.\partial\alpha] + O(M^{-3})$$

Then the power counting and the counting of γ-matrices shows that there is no contribution as $M \longrightarrow \infty$. The same applies to the term [B′,A′] in Eq. (4.53).

APPENDIX G. SYMMETRIC FORMULATION OF THE FAMILY OF ANOMALIES IN $(QED)_4$

As mentioned in Sec. (4.1), chapter IV, our formulation of the family of anomalies of Eqs (4.6) is in terms of a single quantity W_ν , which depends *only* on the regularized axial-vector current. (The reason for this has been given in Sec. (4.1). Following our work [13,14] for a similar family structure in two dimensions, we could think of giving an alternative formulation which is "symmetric" in the vector and axial-vector currents and hence involves *both* the regularized axial *and* vector currents in a symmetric fashion [19]. [For exact definitions see Eqs (G.5) and (G.6) below]. This is the purpose of this appendix.

There is, however, an important difference in two and four dimensions. This arises from the fact that in two dimensions there is really one independent current for,

$$J^A_\mu = - \varepsilon^\nu_\mu \; J^V_\nu$$

and (G.1)

$$J^{AM}_\mu = - \varepsilon^\nu_\mu \; J^{VM}_\nu$$

This facilitates the symmetric formulation of anomalies in two dimensions. The anomaly equations can now be symmetrically expressed in terms of $\partial^\mu \langle J^{AM}_\mu \rangle$ and $\partial^\nu \langle J^{VM}_\nu \rangle$ respectively as the anomaly in either of these is ultimately related to a *single* Green's function $\langle 0|T\,[J^V_\mu(x)\;J^{VM}_\nu(y)]|0\rangle$ on account of Eq (G.1). In four dimensions, J^A_μ and J^V_ν are <u>*independent*</u> currents (giving one of them does not fix the other), and so are J^{AM}_μ and J^{VM}_ν defined as

$$J_\mu^{AM} = \bar{\psi}\ \exp(-\overleftarrow{\not{D}}_a^2/M^2)\ \gamma_\mu\gamma_5\ \exp\ (-\not{D}_a^2/M^2)\ \psi\ , \tag{G-2a}$$

and

$$J_\mu^{VM} = \bar{\psi}\ \exp(-\overleftarrow{\not{D}}_a^2/M^2)\ \gamma_\mu\ \exp\ (-\not{D}_a^2/M^2)\ \psi\ . \tag{G-2b}$$

In four dimensions, the anomaly in QED resides in the triangle diagram, i.e. in the Green's function $\langle 0|T\ [J_\mu^A(x)\ \ J_\nu^V(y)\ J_\lambda^V(z)]|0\rangle$. If the regularization is provided through axial-vector current via Eq (G.2a), one would have $\langle 0|T\ [J_\mu^{AM}(x)\ J_\nu^V(y)\ J_\lambda^V(z)]\ |0\rangle$ and this is what we have dealt with indirectly[1] via $W_\mu[A]$. If one is to regulate the anomalous Green's function via the definition of Eq. (G.2b) of the regularized vector current, one would consider $\langle 0|T\ [J_\mu^A(x)\ \ J_\nu^{VM}(y)\ \ J_\lambda^V(z)]|0\rangle$ suitably symmetrized or $\langle 0|T\ [J_\mu^A(x)\ \ J_\nu^{VM}(y)\ \ J_\lambda^{VM}(z)]|0\rangle$. In QED the difficulty now arises in that while $\langle 0|T\ [J_\mu^{AM}(x)\ \ J_\nu^V(y)\ \ J_\lambda^V(z)]|0\rangle$ can be related to a functional derivative of $W_\mu = \langle J_\mu^{AM}\rangle$ because the vector current is gauged, a similar thing cannot be done within pure QED for $\langle 0|T\ [J_\mu^A(x)\ \ J_\nu^{VM}(y)\ \ J_\lambda^V(z)]|0\rangle$ because the axial-vector current is not gauged and this Green's function cannot be related to $\langle J_\nu^{VM}\rangle$ by taking its functional derivatives. This would imply the need for calculation of the associated functional integral involving product of three currents, a very laborious task indeed. We therefore proceed as below.

We consider the quantity $\langle J_\nu^A(x)\ \ J_\mu^V(y)\rangle_A$ as a functional of A. In providing regularization to this quantity via Eq. (G.2a),

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[1]As seen later, functional derivative of W_ν contains some additional terms also. See Eq. (G.5).

we replaced this as given below :

$$e \langle J^A_\nu(x)\ J^V_\mu(y)\rangle \longrightarrow e \langle J^A_\nu(x)\ J^V_\mu(y)\rangle_{\substack{\text{axial}\\ \text{reg}}}$$

$$\equiv X_{\nu\mu}\ [A,x,y] \equiv -\frac{\delta}{\delta A_\mu(y)} W_\nu[A,x]. \qquad \text{(G.3)}$$

The axial-vector anomaly equation (4.6a) could be formulated in terms of X as

$$\partial^\nu_x X_{\nu\mu}\ [A,x,y] = -\ 2im_o \frac{\delta}{\delta\ A_\mu(y)}\ W_p\ [A]$$

$$+\ \frac{ie^2(1-\beta)}{8\pi^2}\ \varepsilon^{\nu\lambda\sigma}_\mu\ F_{\nu\lambda}(x)\ \partial^x_\sigma\ \delta^4(x-y). \qquad \text{(G.4)}$$

We wish to formulate the vector anomaly equation (4.6b) not in terms of W_ν which depends on regularized <u>axial-vector</u> current but on a quantity symmetric to X, which depends on the regularized <u>vector</u> current. It should be emphasized that, in four dimensions, we are now going to define an *independent* regularization of the triangle diagram and hence have the freedom in defining it, to be used later.

In order to obtain insight into the definition of a quantity symmetric to X, we shall study X first.

Consider ($X \equiv \lim\limits_{M\to\infty} X^M$)

$$\partial^\nu_x X^M_{\nu\mu}\ [\ A;\ x,\ y\ ] = -\frac{\delta}{\delta\ A_\mu(y)}\left[\frac{1}{W\ [A]}\int D\psi\ D\bar\psi\ e^S\ \partial^\nu J^{AM}_\nu(x)\right]$$

$$= \left\{\frac{1}{W\ [A]}\int D\psi\ D\bar\psi\ e^S\left[\partial^\nu J^{AM}_\nu(x)\ J^V_\mu(y) - \frac{\delta}{\delta\ A_\mu(Y)}\ \partial^\nu\ J^{AM}_\nu(x)\right]\right\}_{\text{conn}},$$

(G.5)

where "conn" denotes the connected part. We, now, wish

to *define* a symmetric quantity $\partial^\nu Y^M_{\nu\mu}$ [A;x,y] depending on J^{VM}_ν (x). It is easy to replace the first term in the square bracket on RHS of equation above: It should be replaced by $\partial^\nu J^{VM}_\nu(x) J^A_\mu(y)$. The second term is an axial vector and it should remain so in the definition of Y^M. Hence, we cannot replace $J^{AM} \longrightarrow J^{VM}$ here.

Hence we define in the massless case (which is sufficient to establish the family structure)

$$\partial^\nu Y_{\nu\mu}\ [A;x,y] \equiv \lim_{M\to\infty}\left\{\frac{1}{W\ [A]} \int D\psi\ D\bar{\psi}\ e^S \left[\partial^\nu J^{VM}_\nu(x)\ J^A_\mu(y) - \frac{\delta}{\delta\ A_\mu(y)}\ \partial^\nu\ J^{AM}_\nu(x)\right]\right\}_{conn} + \partial^\nu Y_{o\nu\mu}. \tag{G.6}$$

Here, the term $Y_{o\nu\mu}$ indicates the freedom we have in defining this independent regularization of the triangle diagram. We shall choose Y_o to be independent of 'a' and such that $\partial^\nu Y_{\nu\mu}$ vanishes at a=1. [While this is not *necessary,* it does replace the vector anomaly equation (4.6b) by an equation of the similar form *without* extra local terms.]

We shall, now, indicate as to how $\partial^\nu Y_{\nu\mu}$ can be evaluated. This will be done by relating $\partial^\nu Y_{\nu\mu}$ to $\partial^\nu X_{\nu\mu}$. Consider

$$\left\{\frac{1}{W\ [A]} \int D\psi\ D\bar{\psi}\ e^S\ \partial^\nu J^{VM}_\nu(x)\ \ J^A_\mu(y)\right\}_{conn} \equiv \partial^\nu_x\ Y^{[1]M}_{\nu\mu}\ [A;x,y]. \tag{G.7}$$

In this quantity, we shall replace the representation for γ-matrices by $\gamma_\mu \longrightarrow i\ \gamma_\mu\gamma_5$. The quantity under consideration

becomes[2]

$$\left\{-\frac{1}{W[A]}\int D\psi\, D\bar{\psi}\, e^{S'}\partial^{\nu}J_{\nu}^{AM}(x)\; J_{\mu}^{V}(y)\right\}_{conn} = \partial_x^{\nu}\, Y_{\nu\mu}^{[1]M}\,[A;x,y],$$

$$\text{where}\quad S' = \int d^4x\, \bar{\psi}\,[-\partial\!\!\!/\gamma_5 - ieA\!\!\!/\gamma_5]\psi\;. \qquad (G.8)$$

We notice that $\partial^{\nu}\, Y_{\nu\mu}^{[1]M}$ differs from a similar term in $\partial^{\nu}\, X_{\nu\mu}^{M}$, $\partial^{\nu}\, X_{\nu\mu}^{[1]M}$ in that $S \longrightarrow S'$ and that there is an overall sign change. We now proceed to relate $\partial^{\nu}\, X_{\nu\mu}^{[1]M}$ and $\partial^{\nu}\, Y_{\nu\mu}^{[1]M}$.

Defining

$$-\frac{1}{W[A]}\int D\psi\, D\bar{\psi}\, e^{S}\;\psi(x)\,\bar{\psi}(y) = G(x,y), \qquad (G.9)$$

and

$$-\frac{1}{W[A]}\int D\psi\, D\bar{\psi}\, e^{S'}\psi(x)\,\bar{\psi}(y) = G'(x,y) = i\,G(x,y)\gamma_5$$
$$= -i\,\gamma_5\,G(x,y), \qquad (G.10)$$

We easily see that $\left[f(x) \equiv e^{-x^2/M^2}\right]$

$$\partial_x^{\nu}\, X_{\nu\mu}^{[1]M}\,[A;x,y] = \partial_x^{\nu}\,\mathrm{tr}[G(y,x)\; f(\overleftarrow{D\!\!\!/}_a)\gamma_{\nu}\gamma_5\; f(D\!\!\!/_a)\; G(x,y)\gamma_{\mu}], \quad (G.11a)$$

whereas

$$\partial_x^{\nu}\, Y_{\nu\mu}^{[1]M}\,[A;x,y] = -\partial_x^{\nu}\,\mathrm{tr}[G'(y,x)\; f(\overleftarrow{D\!\!\!/}_a)\gamma_{\nu}\gamma_5\; f(D\!\!\!/_a)G'(x,y)\gamma_{\mu}], \qquad (G.11b)$$

and using (G.10) we see that

$$\partial_x^{\nu}\, X_{\nu\mu}^{[1]M} = \partial_x^{\nu}\, Y_{\nu\mu}^{[1]M}\,. \qquad (G.12)$$

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[2]When the γ-matrices of the theory are changed, the operator $D\!\!\!/_a$ and its eigenfunctions $\phi_n(x)$ change. Hence the measure is also affected in principle. However, the product of Jacobians is 1.

Now, we come to the remaining terms in X and Y. Defining

$$\partial^{\nu} X^{[2]}_{\nu\mu}[A;x,y] = - \lim_{M\to\infty} \frac{1}{W[A]} \int D\psi\, D\bar{\psi}\, e^{S} \frac{\delta}{\delta A_{\mu}(y)} \partial^{\nu} J^{AM}_{\nu}(x)$$

$$= - \lim_{M\to\infty} \partial^{\nu}_{x} \int d^4z\, \delta^4(x-z) \frac{\delta}{\delta A_{\mu}(y)} \left[f(\overleftarrow{\not{D}}_{ax}) \gamma_{\nu}\gamma_5\, f(\not{D}_a) G(x,z) \right]$$

(G.13)

whereas,

$$\partial^{\nu} Y^{[2]}_{\nu\mu}[A;x,y] \equiv \lim_{M\to\infty} (-) \frac{1}{W[A]} \int D\psi\, D\bar{\psi}\, e^{S} \frac{\delta}{\delta A_{\mu}} \partial^{\nu}_{x} J^{AM}_{\nu}(x)$$

$$- \partial^{\nu} Y^{[2]}_{o\nu\mu}[A;x,y]$$

$$= \partial^{\nu} X^{[2]}_{\nu\mu}[A;x,y] - \partial^{\nu} Y^{[2]}_{o\nu\mu}[A;x,y], \qquad \text{(G.14)}$$

where $Y^{[2]}_{o\nu\mu}$ is a-independent and chosen to make $\partial^{\nu} Y^{[2]}_{\nu\mu} = 0$ at a=1.

We thus find that

$\partial^{\nu} Y_{\nu\mu}[A;x,y] = \partial^{\nu} X_{\nu\mu}[A;x,y]$ + a-independent terms to make l.h.s. zero at a = 1. (G.15)

But as seen earlier, $\partial^{\nu} X_{\nu\mu}$ is, in the massless case,

$$\partial^{\nu}_{x} X_{\nu\mu}[A;x,y] = \frac{ie^2(1+a^2)}{8\pi^2} \varepsilon_{\mu}^{\ \nu\lambda\sigma} F_{\nu\lambda}(x)\, \partial^{x}_{\sigma}\, \delta^4(x-y). \qquad \text{(G.16)}$$

Thus Eqs (G.15) and (G.16) yield:

$$\partial^{\nu}_{x} Y_{\nu\mu}[A;x,y] = \frac{ie\,(a^2-1)}{8\pi^2} \varepsilon_{\mu}^{\ \nu\lambda\sigma} F_{\nu\lambda}(x)\, \partial^{x}_{\sigma}\, \delta^4(x-y). \qquad \text{(G.17)}$$

Eqs (G.16) and (G.17) are the analogues of Eqs (4.6) for the massless case. The modification of Eq. (G.16) in the massive case is well known.